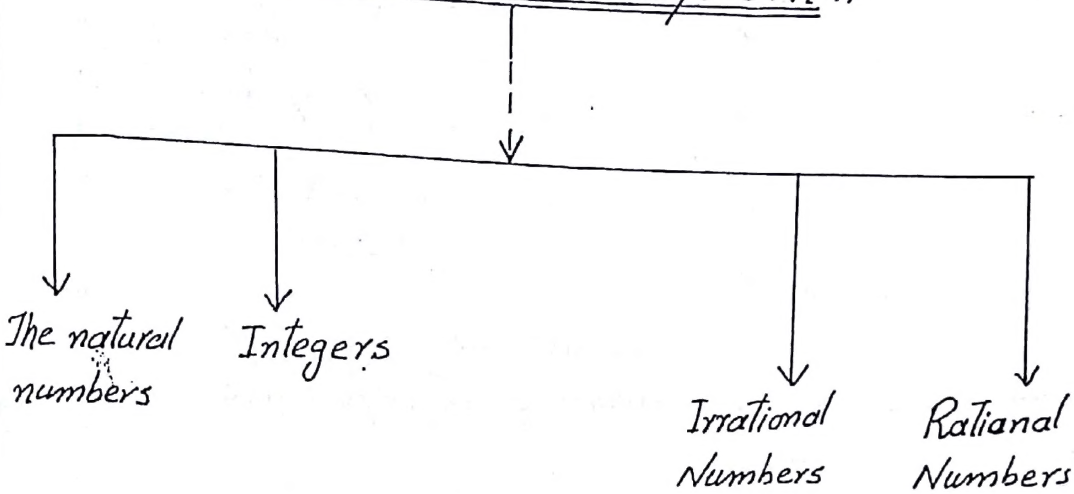
CH - I

Real Number System for M. Sc part - I

CH - 1

Real Number System



Natural Numbers # The simplest numbers are the natural numbers

1, 2, 3, 4 . . . .

Integers

The natural numbers form a subset of larger numbers called the integers

. . . . . . -3, -2, -1 0, 1, 2, 3 . . . .

Rational Numbers

The integers are a subset of a larger class of numbers called rational numbers. Rational numbers are the numbers that can be expressed as the ratio of integers with remainder non-zero. All terminating decimals and recurring decimals are rational numbers.

Irrational Numbers # The numbers which can not be expressed as the ratio of integer with remainder

with remainder non-zero. All non terminating non-recurring decimals are irrational because these can never be expressed as the quotient of integers e.g $\sqrt{3}$, $\sqrt{5}$, $1+\sqrt{2}$, $\sqrt{7}$ π $\cos 19°$, e.

Real Numbers # From above we not that there are two main categories of numbers viz. rational and irrational numbers. The union rational and irrational numbers is called the set of real numbers.

Complex Numbers # Because the square of a real number can not be negative, therefore the equation

$$x^2 = -1$$

has no solutions in the real number system. In the eigtheenth century mathematians remedied the problem by (s) inventing a new number denoted by

$$i = \sqrt{-1}$$

and which defined property $i^2 = -1$. This in turn led to the development of the complex numbers, which are the numbers of the form

$$a+ib$$ where a & b are real numbers.

Note # Every real number (a) is also a complex number because it can be written as

$$a = a+0i$$

Thus the real numbers are subset of complex numbers

Imaginary Numbers#

Those complex numbers that are not real numbers are called imaginary numbers:

The hierachy of numbers is summarized in the fig. below.

Complex number: $a+ib$, where a & b are real numbers and $i = \sqrt{-1}$

Real numbers: rational & irrational numbers such as π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $1+\sqrt{2}$

Rational Numbers: $2/3$, $7/6$, $6/1$

Integers: $\dots -2, -1, 0, 1, 2, \dots$

Natural Numbers: $1, 2, 3, 4 \dots$

Ordered Set#

Let S be a set. An order on S is a relation, denoted by $<$, with following peroperties.

(a)# If $x \in S$, $y \in S$, then one and only of the statements $x < y$, $x = y$, $y < x$ is true

(b)# If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$

An ordered set is a set in which an order is defined. e.g Q is an ordered set, The set of real numbess is an ordered set.

OR

A set S is an ordered set if it satisfies the following Conditions.

(a) $x < y$ or $x = y$ or $y < x$ $\quad \forall x, y \in S$

(b) $y <$ $\quad x < y,\ y < z \Rightarrow x < z \quad \forall x, y, z \in S$

Properties of Order

Order relation "$\leq$" satisfies the following perties

(a) # Reflexivity: $\quad a \leq a \quad \forall a \in R.$

(b) # Anti-symmetry $\quad$ If $a \leq b$ and $b \leq a$, then $a = b$

(c) # Transitivity: $\quad$ If $a \leq b$ and $b \leq c$, then $a \leq c$

(d) # Trichotomy: $\quad$ If a and b are real numbers, then exactly one these three relations holds

$$a < b \quad a = b \quad a > b$$

Examples of ordered Sets

1) # The set $\mathbb{Z}$ of integers.

Let $x, y, z \in \mathbb{Z}$. Then clearly either $x < y$ or $y < x$ or $x = y$.

if $x < y \Rightarrow y - x > 0$

and also if $x < y$, $y < z$, then obviously

$$x < z$$

2) # The set of real numbers

3) # The set of rational numbers

4) # The set of natural numbers

5) # Let $z_1, z_2 \in C$

$$z_1 = x_1 + y_1 i \quad , \quad z_2 = x_2 i + y_2 i$$

which are the points in the Complex (numbers) plane. Since they are representing points, therefore we cannot say that $z_1 < z_2$ or $z_1 > z_2$ or $z_1 = z_2$

Lower Bound of Set

Let E be a non-empty subset of a set S. Then any element $l \in S$ is called an upper bound of E if

$$l \leq x \quad \forall x \in E$$

Upper Bound

A number u is called an upper bound of a non-empty set $E \subseteq S$ if

$$x \leq u \quad \forall x \in E$$

Least Upper bound # (Supremum)

Let E be a non-empty subset of an ordered set S. A number α in S is called least upper bound for E if

(i): α is an upper bound for E

(ii): If a is any other upper bound of E, then

$\alpha \leq a$. i.e.

α is the upper bound of E and is not greater than any other upper bound of E. The least upper bound is called supremum and is denoted by $lub E = Sup E$.

Note: A set having at least one upper bound is said to be bounded above

Greatest Lower Bound # (Infimum)

Let E be a non-empty subset of an ordered set S. A number β in S is called the greatest lower bound for E if.

(a) # β is a lower bound for S

(b) # If b is any other lower bound for E, then

$b \leq \beta$ i.e β is not less than any other bound (lower) of E.

Thus the greatest ~~upper~~ lower bound is an ~~upper~~ lower bound of E which is not less than any other lower bound.

It is denoted by $Inf E = lub E = \inf\limits_{x \in E}$

Note # A set having at least one lower bound is said to be bounded below.

Bounded Set

A non-empty subset E of an ordered set is bounded if it has at least one upper bound and at least one lower bound i.e

E is both bounded above and bounded below.

OR

E is called bounded if there exist two number l & u in S such that

$$l \leq x \leq u. \quad \forall x \in E$$

Note (1): If E is unbounded above, we write $\sup E = +\infty$ and if E is unbounded below we write $\inf E = -\infty$

2) # If E is finite set we also use notations $\max E$ and $\min E$ for $\sup E$ and $\inf E$.

Examples

1) # Let $A = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ be a subset of $\mathbb{Q}$. Then 0 and all rational numbers less than 0 are lower bounds of A. Hence

$$\inf A = 0 \in \mathbb{Q}.$$

$$\text{But } \inf A \notin A.$$

Similarly 1 and all rational numbers greater than 1 are upper bounds for A. Hence

$$\sup A = 1 \in \mathbb{Q}$$

$$\text{Also } \sup A \in A.$$

2) #

$$\text{lub}(0, 1) = 1 = \text{lub}[0\ 1]$$

$$\text{glb}(0\ 1) = 0 = \text{glb}[0\ 1]$$

Theorem # The least upper bound (glb) of a non-empty subset X of an ordered set S or ordered field is unique if it exists.

Proof # Let on the contrary

$$\alpha_1 = \sup X \quad \& \quad \alpha_2 = \sup X \quad \text{i.e.}$$

X has two supremum.

$\because$ α_1 is an upper bound of X and $\alpha_2 = \sup X$

$\therefore$ By definition of Supremum

$$\alpha_2 \leq \alpha_1 \longrightarrow ①$$

Similarly since $\alpha_1 = \text{Sup}\, X$ and α_2 is an upper bound of X, therefore

$$\alpha_1 \leq \alpha_2 \longrightarrow ②$$

① & ② are possible only if

$$\alpha_1 = \alpha_2$$

Hence Supremum if it exist is unique.

Completeness Property (OR Least-Upper bound Property of and ordered field)

An ordered set S has completeness property or least-upper bound property if every non-empty subset of S which is bounded above has least upper bound in S. e.g The $\mathbb{Q}$ does not have the least upper bound property while the set of Real numbers has this property which will be proved Later in properties of $\mathbb{R}$.

Complete Ordered Set#

An ordered set is said to be Complete if it has least-upper bound property i.e every non-empty subset of it which is bounded above has the supremum in it.

Remarks# We shall now prove that there is a close relation between greatest lower bounds and least upper bounds and that every ordered set with the least upper bound property also has the greatest lower bound property.

Theorem# Suppose that S is an ordered set with the least upper bound property and B is a non-empty subset of S which is bounded below. Let L be the set of all lower bounds of B. Then

$\alpha = \sup L$ exists in S and

$\alpha = \inf B$.

In particular, $\inf B$ exists in S.

OR

Every ordered set with the least upper bound property also has the greatest lower-bound property. OR If a non-empty set of a complete ordered set is bounded below, then it has an infimum.

Proof# Since B is bounded below, therefore L is non-empty

$\because$ L is the set of lower bounds of B in S

Therefor $L = \{y : y \in S \text{ and } y \leq x \quad \forall x \in B\}$

Thus every element of B is an upper bound of L. and hence L is bounded above

Now L is non-empty bounded above subset of S and S has l.u.b property. Therefore L has sup in S.

Let $\sup L = \alpha$

If $\gamma < \alpha$, then by definition of sup γ is not an upper bound of L.

Hence $\gamma \notin B$

If $\alpha < \beta$, then $\beta \notin L$ because α is an upper bound of L. In other words α is a lower bound of B but β is not if $\beta > \alpha$.

Consequently $\alpha = \inf B$

Remarks: From here we note that for ordered set S with least upper bound property and for every non-emply bounded below subset B of S we can alway design the set of all lower bounds L such that $\sup L = \inf B$

Also for every ordered set S with (least upper bound) greatest lower bound property and for every non-empty bounded above subset A of S we can design the set ~~of all upper bounds U such that~~

$$\operatorname{Inf} U = \operatorname{Sup} A.$$

Thus we can say an ordered S has least upper bound property iff it has greatest lower bound property.

We have the following proposition..

Proposition # Let L & U be non-empty subsets of an ordered set S with $S = L \cup U$ and

$$l \leq u \quad \forall l \in L \text{ and } \forall u \in U.$$

Then either L has a greatest element or U has a least element.

Theorem # Let 'A' be a subset of an ordered set S such that 'A' is bounded below, then

(i) $\operatorname{Inf} A = -\operatorname{Sup}(-A)$ $\quad \operatorname{Sup}(-A) = -\operatorname{Inf} A$

where $-A = \{-x : x \in A\}$ $\quad$ (ii) $\operatorname{Inf}(-A) = -\operatorname{Sup} A$

Proof # Since 'A' is bounded below, $\inf(A)$ will exist.

Let $\inf(A) = g$.

Then $g \leq x \quad \forall x \in A \longrightarrow (1)$

But $g + \epsilon > x.$ for some x in A

Now $g \leq x \quad \forall x \in A$

$-g \geq -x \quad \forall x \in A$

or $-x \leq -g \quad \forall x \in A$

$\Rightarrow -g$ will be the upper bound of $-A$

Also $-g - \epsilon < -x$

or $-x > -g - \epsilon$

$\therefore -g$ is l.u.b of $-A$

Hence $\operatorname{Inf}(A) = g = -(-g) = -\operatorname{Sup}(-A)$

Question # Let E be a non-empty set of an ordered set S. Then

$$\operatorname{Inf} E < \operatorname{Sup} E.$$

Proof # Let α, β be inf & sup of E. Then

$$\alpha \leq x \quad \forall x \in E$$

$$\& \quad x \leq \beta \quad \forall x \in E$$

$$\Rightarrow \alpha \leq x \leq \beta \quad \forall x \in E$$

$$\Rightarrow \alpha < \beta \quad \text{(Proved)}.$$

Theorem # Let E be a non-empty bounded set in an ordered set S. Then

(a) # For a fixed c in S^E define $c+S^E = \{c+x : x \text{ in } S^E\}$

(b) # Prove That $\operatorname{Sup}(c+S^E) = c + \operatorname{Sup} S^E$ and

$$\operatorname{Inf}(c+S^E) = c + \operatorname{Inf} S^E$$

(b). # For a fixed $c > 0$, defined $cE = \{cx : x \text{ in } E\}$

Prove that $\operatorname{Inf}(cE) = c \operatorname{inf} E$

and $\operatorname{Sup}(cE) = c \operatorname{Sup} E$

(c) # In part (b), $c < 0$, show that $\operatorname{Sup}(cE) = c \operatorname{inf} E$ and

$$\operatorname{inf}(cE) = c \operatorname{Sup}(E)$$

Proof # (a) Let $\alpha = \operatorname{Sup} E$.

Then $x \leq \alpha \quad \forall x \in E$

and $\alpha - \epsilon < x$ for some x in E

Now $c + x \leq c + \alpha \quad \forall x \in E$

$\Rightarrow c + \alpha$ is an upper bound of E

Again $c + \alpha - \epsilon < c + x$ for some x in E

Hence $\operatorname{Sup}(c+E) = c + \alpha$

$$= c + \operatorname{Sup} E$$

Similarly it can be proved that

$$\operatorname{Inf}(c+E) = c + \operatorname{Inf} E$$

(b)# $\alpha = Inf(E)$

Then $\alpha \leq x \quad \forall x \in E$

and $\alpha + \epsilon > x$ for some x in E

Now since $c > 0$, therefore

$$c\alpha \leq cx \quad \forall x \in E$$

$\Rightarrow$ $c\alpha$ is a lower bound of cE

Again $c\alpha + c\epsilon > cx$ for some x in E

$$\Rightarrow Inf(cE) = c\alpha$$
$$= c\,Inf(E)$$

Similarly it can be proved that

$$Sup(cE) = c\,Sup(E)$$

(c) Let $\alpha = Sup(E)$

Then $x \leq \alpha \quad \forall x \in E$

and $\alpha - \epsilon < x$ for some x in E

$\because c < 0$

$$\therefore \quad cx \geq c\alpha \quad \forall x \in E$$

$\Rightarrow$ $c\alpha$ is a lower bound for cE

Also $c\alpha + c\epsilon > cx$ for some x in E

$$\Rightarrow Inf(cE) = c\alpha$$
$$= c\,Sup(E)$$

__Theorem#__ Let A be a non-empty subset of an ordered set S and let u, v are numbers in S. Then

(a)# If A is bounded above, Then $u = Sup\,A$ iff u is u is an upper bound for A and for every $\epsilon > 0$, there exists an x in A, such that $u - \epsilon < x \leq u$.

(b)# If A is bounded below, Then $v = Inf(A)$ iff v is a lower bound for A and for every $\epsilon > 0$, there exists an x in A such That

$$v \leq x < v + \epsilon$$

__Proof#__ (a) Let $u = Sup\,A$. Then

$$x \leq u \quad \forall x \in A$$

If there were no point x of A such that

$$\mu - \epsilon < x \leq \mu$$

Then $\mu - \epsilon$ will be an upper bound for A, which a contradiction because μ is least upper bound for A. Consequently there must be an x in A such that

$$\mu - \epsilon < x \leq \mu$$

Conversely suppose that μ is an upper bound for A with the property that for every $\epsilon > 0$ $\exists$ a point x in A such that

$$\mu - \epsilon < x \leq \mu$$

we are to show that μ is L.u.b. For this let M be any upper bound for A and on the contrary

$$M < \mu$$

Let $\epsilon = \mu - M$, then $\epsilon > 0$

Hence there must be a point x in A such that

$$\mu - \epsilon < x \leq \mu$$

But $\mu - (\mu - M) < x \leq \mu$.

$$M < x \leq \mu$$

This is a contradiction because M is an upper bound of A. Thus

$$\mu \leq M$$

and hence μ is least upper bound for A.

(b) # Similarly can be proved. Prove it.

Relations & Equivalences

Relation # Two given quantities x and y may be "related" to each other in many ways as in $x = y$, $x \in y$, $x \subset y$ or for numbers $x < y$.

In general we say that R denoteds a relation if, given x and y either x stands in the relation R to y (written $x R y$) or x does not stand in the relation R to y. A relation R is said to a relation on a set X

if $x R y \Rightarrow x \in X$ and $y \in X$. If R is a relation on a set X we define the graph of R to be the set

$$\{(x, y) : x R y\}.$$

Since we consider two relations R and S to be the same if $(x R y) \Leftrightarrow (x S y)$, each relation on set X is uniquely determined by its graph. and Conversely each subset of $X \times X$ is the graph of some relation on X. Thus we may identitify a relation on X with its graph and define a relation to be a subset of $X \times X$

<u>Remarks</u> # In many formalized treatment of set theory a relation is in general defined simply as a set of ordered pairs. It is should be pointed out, however, that there is a difficulty in this approach in that $'='$, $'\in'$, and $\subset$ are no Longer relations. Therefore It comes out that relations are not necessarily sets of ordered pairs.

Symmetric Relation

A relation. R is said to be symmetric on X if

$$x R y \Longrightarrow y R x \qquad \forall x, y \in X$$

e.g If $X = \{a, b, c\}$
Then relation $R = \{(a, b)\ (b, a), (b\ c), (c, b)\}$
is a symmetric Relation on X

Reflexive Relation

A Relation. R is said to be Reflexive on the set X if

$$x R x \qquad \forall x \in X$$

e.g If $X = \{a\ b, c\}$
Then $R_1 = \{(a, a), (b, b), (c, c)\}$ is reflexive on X
It is also called diagonal relation on set X.

Transitive Relation

A relation R is said to be transitive on a set X if.

$$xRy \text{ and } yRz \Rightarrow xRz \quad \forall x,y,z \in X$$

e.g Relation of equality $(=)$ and $(<)$ are transitive relations on the set of real numbers.

Antisymmetric Relation

A relation R is said to be anti-symmetric on a set X if whenever

xRy and yRx, then $x=y$

Equivalence Relation

A relation which is transitive, reflexive, and symmetric on set X is called an equivalence relation on set X or simply an equivalence on X.

Equivalence Classes of Set & Their Characteristics

Suppose that R is an equivalence relation a set X. For a given $x \in X$, let E_x or C_x be the set of all elements of X equivalent to x (all elements in relation with x under R) i.e

$$E_x = \{y : yRx\}$$

Then E_x is called an equivalence set or class of X under R.

Characteristics of equalence classes of X are

1)# Any element of X which is equivalent to any element of E_x (not necessarity x) is itself an element of E_x

Proof # Let $y \in E_x$

Then for any element z of X, equivalent y we have zRy but yRx and R is transitive

$\Rightarrow \quad z R x$

$\Rightarrow \quad z \in E_x$ (proved)

2) # For any two elements x and y of X, the sets E_x & E_y are either identical (if $x R y$) or disjoint (if $x \not R y$)

Proof # Let E_x and E_y not be identical. We are to prove that $E_x \cap E_y = \varphi$.

Let on the contrary $E_x \cap E_y \neq \varphi$ but

$$z \in E_x \cap E_y$$

Then $z \in E_x$ & $z \in E_y$

$\Rightarrow z R x$ & $z R y$

$\Rightarrow x R z$ & $z R y \longrightarrow$ (A) $\because$ (R is symmetric)

$\Rightarrow x R y$ $(\because T$ is transitive)

$\Rightarrow$ Every element equivalent to x will also be equivalent to y. Equivalently every element of E_x is in E_y and hence

$$E_x \subseteq E_y \longrightarrow (1)$$

Again from (A)

$z R x$ & $y R z$ $\because$ (R is symmetric)

$\Rightarrow y R x$

$\Rightarrow$ Every element equivalent to y will also be equivalent to x. Equivalently every element of E_y is in E_x and hence

$$E_y \subseteq E_x \longrightarrow (2)$$

By (1) & (2)

$$E_x = E_y$$

which is a contradiction to our supposition that E_x & E_y are not identical. Thus $E_x \cap E_y = \varphi$

Conversely let E_x & E_y are not disjoint

we are to prove that E_x & E_y are identical.

On the contrary E_x & E_y be not identical.

Then since $E_x \cap E_y \neq \varphi$

Therefore as above $E_x = E_y$ i.e E_x and E_y are identical.

Remarks # The sets in collection $\{E_x : x \in X\}$ are called equivalence sets or classes of X under equivalence relation R ($\equiv$). Thus X is disjoint union of the equivalence classes under the equivalence relation R i.e X is partioned by the equivalence classes und R or R partions X into dijoint equivalence classes.

The Collection of equivalence classes under an equivalence relation R is called qoutient of X with respect to R and is sometimes denoted by X/R.

The mapping $x \longrightarrow E_x$ is called the natural mapping of X onto X/R

A binary operation on a set X is a mapping from $X \times X$ to X. We say that an equivalence relation R on set X is Compatible with a binary operation $(+)$ if xRx' and $yRy' \Rightarrow (x+y) R (x'+y')$

In this case $(+)$ defines an operation on the quotient $Q = X/R$ as follows

If E & F belong to Q, choose $x \in E$, $y \in F$ and define $E+F$ to be $E_{(x+y)}$. Since R is an equivalence relation, $E+F$ depends only on E and F and not on the choice of x and y.

Partial Ordering Relation

A relation $\prec$ is said to be a partial ordering of a set X (or to partially order X) if it is tansitive and antisymmetric on it (X).

Thus $\leq$ is a partial ordering on the set of real numbers and $\subset$ is a partial ordering on $P(X)$, power set of X.

Linear Ordering (ordering)

A partial ordering $\prec$ on a set X

is said to be a linear ordering (or simply ordering) of X if for any two elements of for any two elements x & y of X we have

either $x \angle y$ or $y \angle x$

Thus $\leq$ linearly orders the set of real numbers. while $\subset$ is not a linear ordering on $P(X)$.

Remarks # If $\angle$ is a partial order on X and if $a \angle b$, we often say that a precedes b or that b follows a. Sometimes we say that a is less than a or b is greater than a.

First element OR the smallest Element in a Set

If $E \subset X$ ($\subset$ is partial order on X), then an element $a \in E$ is called 1st element in E or the smallest element in E if whenever $x \in E, x \neq a$, then we have

$$a \angle x$$

Similarly for last (or the largest) element of E.

Minimal Element of E

An element $a \in E$ is called a minimal element of E if there is no $x \in E$ with $x \neq a$ such that $x \angle a$.

Similarly for maximal elements.

Note # 1)# It should be observed that if a set has a smallest element, then that element is a minimal element.

2)# If $\angle$ is a linear ordering, a minimal element is a least element but in general it is possible to have minimal elements which are least elements.

Reflexive Partial Order

Our definition of partial order

makes no assertion about the possibility or necessity that under a partial order $\prec$

$$x \prec x$$

If we have $x \prec x$ for all x, $\prec$ is called reflexive partial order: $\leq$ is reflexive partial order on set of real numbers.

Strict Partial Order#

A partial order $\prec$ on set X is celled strict partial order if we neve have $x \prec x$.

Thus $<$ is a strict partial order for real numbers

Note# To any partial order $\prec$ there is associated a unique strict partial order and a unique reflexive partial order that agree with $\prec$ for all (x, y) with $x \neq y$. If $\prec$ is any partial order we use $\preceq$ for the associated reflexive order.

Housdorff Maximal Principle#

Let $\prec$ be a partial ordering on a set X. Then there is a maximal linearly ordered subset S of X i.e a subset S of X which is linearly ordered by $\prec$ and has the property that $S \subset T \subset X$ and T is linearly ordered by $\prec$, then $S = T$ i.e there is no supper set of S in X which is linearly ordered

Remarks: This principle is equivalent to the axiom of choice and is often more convenient to apply.

Axiom of Choice#

Let $\mathcal{C}$ be any collection of non-empty sets

Then there is a function F defined on $\mathcal{C}$ which assigns to each set $A \in \mathcal{C}$ an element $F(A)$ in A.

Remarks# The function F is called choice function. If there are only a finite no of sets in $\mathcal{C}$, there is no difficulty in choosing for each of the sets A in $\mathcal{C}$ and element in A but we need the choice axiom in case the collection $\mathcal{C}$ is infinite. If the set in $\mathcal{C}$ is disjoint, we may think of the axiom of choice as asserting the possibility of selecting a "parliment" consisting of member from each of the sets in $\mathcal{C}$

Bertrand Russell prefers to call the axiom of chice the multiplicative axiom.

Well Ordering#

A strict linear ordering $<$ on a set X is called a well ordering for X or is said to well order X if every non-empty subset of X contains a first element.

Thus if $X = N$ (set natural numbers) and $<$ to means less than, then N is well ordered by $<$. On the other hand, the R of all real nos is not well ordered by the relation "less than".

The following principle clearly implies the axiom of choice and can be shown equivalent to it

Well ordering Principle#

Every set X can be be well ordered i.e there is a relation $<$ which well orders X.

Existence Theorem# (For the set of Real numbers)

There exists an ordered fied R which has the least upper bound property.

Moreover, R contains as a 'a sub field.

The statement means that $Q \subset R$ and that operations of addition and multiplication in R, when applied to members of Q

coincide with the usual operations on rational numbers. Also the +ve rational numbers are +ve elements of R.

The members of R are called real numbers

Axioms of Real Numbers

Assume the set R of real numbers, the set P of +ve real numbers. The axioms of real numbers fall into three groups

1): Algebric or Arithmetical properties (addition, subtraction multiplication, division (except by 0) to produce more real numbers. These are also called field axioms

2): Order axioms or order properties.

3):- Completeness axiom or Least upper bound property:

The Field Axioms

A_1: There is a binary operation called addition and denoted by '+' such that

$$x+y \in R \quad \forall x, y \in R.$$

i.e addition is defined on R.

A_2: Addition is associative i.e

$$(x+y)+z = x+(y+z) \quad \forall x, y, z \in R$$

A_3:- Additive Identity exists in R i.e $\exists\, 0 \in R$ such That

$$x+0 = 0+x = x \quad \forall x \in R$$

A_4: Additive Inverses exist in R i.e for each $x \in R$ there exists $y \in R$ such that

$$x+y = y+x = 0$$

The additive inverse of each element is unique and for each $x \in R$, it is denoted by $-x$

we define $x - z = x + (-z) \quad \forall x, z \in R$

A_5: Addition is Commutative i.e

$$x+y = y+x \quad \forall x, y \in R$$

Any mathematical system which satisfied these axioms

is called a Commutative (abelian) group under '+'. Thus $(R, +)$ is an abelian group.

A6: There is a binary operation called multiplication defined in R i.e.

$$x \cdot y\ (xy) \in R \qquad \forall x, y \in R$$

A7: Multiplication is associative i.e

$$(xy)z = x(yz) \qquad \forall x, y, z \in R$$

A8: Multiplication is commutative. i.e

$$xy = yx \qquad \forall x, y \in R$$

A9: A multiplicative identity exists i.e there exists a real number 1 different from 0 such that

$$1 \cdot x = x \cdot 1 = x \qquad \forall x \in R$$

A10: Multiplicative inverses exist for non-zero real numbers i.e for each non-zero x in R, $\exists\ y \in R$ such that

$$xy = 1 = yx$$

The next axiom links the operations of addition and multiplication.

A11: Multiplication distributes over addition. i.e

$$y(x+z) = yx + yz$$
$$(y+x)z = yz + xz \qquad \forall x, y, z \in R$$

2): Order Axioms

There is a subset P of R called positive real numbers which satisfies the following

(i) $x + y \in P$ $\forall x, y \in R$ i.e operation of addition is defined on P

(ii) $xy \in P$ $\forall x, y \in P$ i.e operation of multiplication is defined in P

(iii) $x \in P \Rightarrow -x \notin P$

(iv) $x \in R \Rightarrow x = 0$ or $x \in P$ or $-x \in P$ i.e

If x is in R, exactly one the following is true

$$x \in P \quad \text{or} \quad x = 0 \quad \text{or} \quad -x \in P$$

i.e R is partitioned into sets $\{0\}$, P and $\{-P\}$, the set of inverses of all elements in P under addition

OR

We have a relation $<$ which establishes an ordering among the real numbers and which satisfies following axioms

(1) $x < y$ & $z < w \Rightarrow x+z < y+w$ equivalent to (i) above

(2) $0 < x < y$ & $0 < z < w \Rightarrow xz < yw$ equivalent to (ii) " "

(3) Exactly one of the relation holds $x = y$, $x < y$, $x > y$ holds
$\Rightarrow$ of any two different numbers one must be larger
(equivalent to iv above)

(4) If $x > y$, $y > z$, then $x > z$

Remarks Since the relation $<$ is transitive we see that real numbers are linearly ordered by $<$. Thus the real numbers are an ordered field.

Geometric Representation of Real Numbers:

~~numbers can be represented~~ The real numbers can be represented geometrically as points on line (the real axis). A point is selected to represent 0 and another point to represent 1 and these points determine the scale. Then each point on real axis corresponds to one and only one real number and conversely, each real number is represented by a single. So there is a one-to-one correspondence between the points on real axis and real numbers.

Proposition# The axioms of addition imply the following Conditions

(a) If $x+y = x+z$, then $y = z$ (Cancellation Law)

(b) If $x+y = x$, then $y = 0$

(c) If $x+y = 0$, then $y = -x$

(d) If $-(-x) = x$

Proof # (a) $y = y+0 = y+x+(-x)$.

$$
\begin{aligned}
y &= y+0 && \text{(Identity Law of R)} \\
&= y+x+(-x) && \text{(existence of the inverse)} \\
&= x+y+(-x) && \text{(Commutative Law)} \\
&= x+z+(-x) && \text{(given)} \\
&= z+x+(-x) && \text{(Commutative Law)} \\
&= z+[x+(-x)] && \text{(Associative Law)} \\
&= z+0 && \text{(Inverse Law)} \\
y &= z && \text{(Identity Law)}
\end{aligned}
$$

(b) # from the relation in (a)

If $x+y = x+z \Rightarrow y = z$

putting $z=0$

Then $x+y = x+0$

$x+y = x$

$y = 0$

(c) # Given that

$x+y = x$

$x+y = x+0$ (Identity Law)

$\Rightarrow y = 0$ by (a)

(d) # $\because -x+x=0$

$x+(-x) = 0$ Comutative Law

$\Rightarrow -x+x = 0$

$-x-(-x) = 0$

$x - x - (-x) = x+0$

$(x-x)-(-x) = x$

$0-(-x) = x$

$-(-x) = x$

Proposition# The axioms for multiplication imply the following.

(a) If $x \neq 0$ and $xy = xz$, then $y = z$

(b) If $x \neq 0$ and $xy = x$, then $y = 1$

(c) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$

(d) If $x \neq 0$, then $\frac{1}{(1/x)} = x$ or $(x^{-1})^{-1} = x$

Proof# (a)#

$$\begin{aligned} y &= 1 \cdot y && \text{(Identity Law)} \\ &= (x x^{-1}) y && \text{(Inverse)} \\ &= (x^{-1} x) y \\ &= x^{-1}(xy) && \text{associative Law} \\ &= x^{-1}(xz) && \because xy = xz \text{ given.} \\ &= (x^{-1}x) z && \text{associative Law} \\ &= 1 \cdot z && \text{Inverse.} \\ &= z && \text{Identity} \end{aligned}$$

(b)# Given that

$$\begin{aligned} &xy = x \\ \Rightarrow\ &xy = x \cdot 1 && \text{Identity} \\ \Rightarrow\ &y = 1 && \text{by (a)} \end{aligned}$$

(c)#

$$\begin{aligned} &xy = 1 \\ &xy = x\left(\tfrac{1}{x}\right) && \text{Inverse.} \\ \Rightarrow\ &y = \tfrac{1}{x} && \text{by (a)} \end{aligned}$$

(d)# We know that

$$\begin{aligned} &x x^{-1} = 1. \\ \Rightarrow\ &(x^{-1})(x^{-1})^{-1} = 1 \\ &x x^{-1} (x^{-1})^{-1} = x \cdot 1 && \text{associative Law \& identity Law} \\ &1 \cdot (x^{-1})^{-1} = x. && \text{Inverse.} \\ \Rightarrow\ &(x^{-1})^{-1} = x. && \text{Identity} \end{aligned}$$

Proposition # The field axioms imply the following statements for all $x, y, z \in R$.

(a) $0x = 0$

(b) If $x \neq 0$, $y \neq 0$, then $xy \neq 0$

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

Proof # (a) #

$$0\cdot x + 0\cdot x = (0+0)x \qquad \text{distributive law}$$
$$= 0x \qquad \text{Identity}$$
$$\Rightarrow 0\cdot x + 0\cdot x = 0\cdot x$$
$$\Rightarrow 0x = 0 \qquad \text{If } x+y = x \text{, then } y = 0$$

(b) # Assume that $x \neq 0$, $y \neq 0$ but $xy = 0$

As $x x^{-1} y y^{-1} = 1\cdot 1 = 1 \longrightarrow (A)$

Also
$$x x^{-1} y y^{-1} = xy\, x^{-1} y^{-1} \qquad \text{Commutative law}$$
$$= 0\, x^{-1} y^{-1} \qquad \text{by assumption } xy = 0$$
$$= 0 \longrightarrow (B) \qquad \because 0x = 0$$

from (A) & (B) we have

$1 = 0$ which is a contradiction.

Thus $xy \neq 0$

(c) # $(-x)y = -(xy) = x(-y)$

$\because 0y = 0$
$$\Rightarrow (x - x)y = 0 \qquad \text{Inverse.}$$
$$xy + (-x)y = 0 \qquad \text{distributive law}$$
$$-(xy) + xy + (-x)y = -(xy) + 0 \qquad \text{Inverse}$$
$$0 + (-x)y = -(xy) + 0$$
$$(-x)y = -(xy) \longrightarrow (1) \qquad \text{Identity}$$

Again
$$0 = x\cdot 0$$
$$= x(y - y) \qquad \text{Inverse.}$$
$$= xy + x(-y) \qquad \text{Dist. Law}$$
$$-(xy) = -(xy) + (xy) + (x(-y))$$
$$-(xy) = x(-y) \longrightarrow (2) \qquad \text{Identity}$$

By (1) & (2)

$(-x)y = -(xy) = x(-y)$ proved

(d) # $(-x)(-y) = xy$

$(-x)(-y) = -[x(-y)]$ by (c) above

$= -(-xy)$ by (c) above

$= xy$ $\because -(-x) = x$.

Proposition # The order axioms of R imply the following. In fact these are true for any ordered field.

(a) If $x > 0$, then $-x < 0$ and vice versa

(b) If $x > 0$ and $y < z$, then $xy < xz$

(c) If $x < 0$ and $y < z$, then $xy > xz$

(d) If $x \neq 0$ then $x^2 > 0$. In particular $1 > 0$

(e) If $0 < x < y$ then $0 < 1/y < 1/x$.

Proof (a) # $x > 0$

$x > y$ $\Rightarrow x+z > y+z$

$(-x) + x > (-x) + 0$

$0 > (-x) + 0$ Inverse

$0 > -x$ Identity

or $-x < 0$

Again $x < 0$

$-x + x < -x + 0$

$0 < -x$

$-x > 0$

i.e. If $x > 0$, $-x < 0$ & if $x < 0$, $-x > 0$

(b) * $y < z$.

$-y + y < -y + z$ existence of inverse

$0 < z - y$

Since $x > 0$

$x(z-y) > 0$ If $x > 0$, $y > 0$ $xy > 0$

$xz - xy > 0$ distributive law

$xz - xy + xy > 0 + xy$

$\Rightarrow xz > xy$

(c) $y < z$

$\Rightarrow -y+y < -y+z$

$\Rightarrow z-y > 0$

As $x < 0$ $\quad -x > 0$ $\qquad \because x < \Rightarrow -x > 0$

$\Rightarrow -x(z-y) > 0$

$\Rightarrow -xz + xy > 0$ $\qquad$ distributive laws

$\Rightarrow -xz - (x)(-y) > 0$

$\Rightarrow -xz + xy > 0$ $\qquad$ as $-(-x) = x$

$xz - xz + xy > 0 + xz$

$0 + xy > 0 + xz$

$xy > xz$

(d) # Case I: when $x > 0$

$x \cdot x > 0$

$\Rightarrow x^2 > 0$

case II : when $x < 0$

$\Rightarrow -x > 0$

$\Rightarrow (-x)(-x) > 0$

$-(-x^2) > 0 \quad \Rightarrow x^2 > 0$

and if $x > 0$

$\Rightarrow 1/x > 0$

$\Rightarrow x \cdot 1/x > 0 \quad \Rightarrow 1 > 0$

(e) # $x > 0 \Rightarrow \frac{1}{x} > 0$

$y > 0 \Rightarrow 1/y > 0$

Hence $(\frac{1}{x})(\frac{1}{y}) > 0$

Given that $0 < x < y$

$0 \cdot \frac{1}{x} \cdot \frac{1}{y} < x \cdot \frac{1}{x} \cdot \frac{1}{y} < y \cdot \frac{1}{x} \cdot \frac{1}{y}$

$0 < (x \cdot \frac{1}{x}) \cdot \frac{1}{y} < y \cdot \frac{1}{y} \cdot \frac{1}{x}$ $\qquad$ Commutative

$0 < 1 \cdot \frac{1}{y} < 1 \cdot \frac{1}{x}$ $\qquad$ Inverse

$0 < \frac{1}{y} < \frac{1}{x}$ $\qquad$ Identity

~~Completeness Axiom of Real Numbers~~ #

Theorem# The additive identity of Real numbers is unique.

Proof: Let there exists $0' \in R$ such that

$$x + 0' = x \qquad \forall x \in R$$

Then $\quad 0 + 0' = 0 \qquad$ by property of $0'$

Also $\quad 0' + 0 = 0'$

so $\quad 0 = 0 + 0' = 0' + 0 = 0' \qquad$ Commutative Law

$$0 = 0'$$

$\Rightarrow$ Identity is unique.

3: Completeness Axiom of Real Numbers

Every non-empty subset of real numbers which is bounded above has a least upper bound in R

As a consequence of this axiom, it follows that every non-empty set of real numbers which is bounded below has an infimum R

Theorem# A subset of real numbers which is bounded below has a greatest lower bound.

Proof# Let X be the set of real numbers which is bounded below and let Y be the set of lower bounds for X. Let $c \in X$. Then

$$y \le c \qquad \forall y \in Y$$

Thus Y is bounded above. By the least upper bound axiom Y has a least upper bound (a). We show that (a) is the greatest lower bound of X

Let $x \in X$. Then $y \le x \qquad \forall y \in Y$

Thus x is an upper bound for Y

Since 'a' is the least upper bound for X, we have

$$a \leq x$$

$\Rightarrow$ a is a lower bound for X.

Let 'b' be any lower bound for X. Then $b \in Y$.
Hence by definition of least upper bound

$$b \leq a$$

$\Rightarrow$ a is the greatest lower bound of X

OR

Let $B = \{-x : x \in X\}$

Then B is non-empty set because X is non-empty.

$\because$ X is bounded below

$\therefore \exists$ an element 'a' of R such that

$$a \leq x \quad \forall x \in X$$

But then $-a \geq -x \ \forall x \in X$ or for all $-x \in B$

$\Rightarrow$ $-a$ is an upper bound for B and B is bounded above

By least upper bound property B will have least upper bound.

Let $\mu = \sup B$

Then $-x \leq \mu \quad \forall -x \in B \quad \forall x \in X$

$\Rightarrow \quad x \geq -\mu \quad \forall x \in X$

$\Rightarrow$ $-\mu$ is a lower bound for X

Let l be any other bound for X. Then

$$l \leq x \quad \forall x \in X$$

$\Rightarrow \quad -x \leq -l \quad \forall -x \in B$

$\Rightarrow$ $-l$ is an upper bound for B

But then $\mu \leq -l \quad \because \mu = \sup(B)$

$\Rightarrow \quad -\mu \geq l$

$\Rightarrow$ $-\mu$ is the greatest lower bound for X

Problem # Show that $\sqrt{2}$ is not a rational number.

Sol: Assume that $\sqrt{2}$ rational number
Then $\exists$ integers p & q such that
$$\sqrt{2} = p/q \qquad q \neq 0 \longrightarrow (1)$$
Suppose that p and q have no the common factor other than 1 i.e fraction p/q is reduced to the lowest terms. It means p & q are both not ~~simultaneously~~ even.

Squaring (1)
$$(\sqrt{2})^2 = (p/q)^2$$
$$\Rightarrow 2 = \frac{p^2}{q^2}$$
$$\Rightarrow p^2 = 2q^2 \longrightarrow (1)$$
$\Rightarrow p^2$ is an even integer
Since square of an odd integer is odd, therefore p must be an even integer.
$$\text{Let } p = 2m \qquad m \in \mathbb{Z}$$
putting in (1)
$$(2m)^2 = 2q^2$$
$$\Rightarrow 4m^2 = 2q^2$$
$$\Rightarrow q^2 = 2m^2$$
$\Rightarrow q^2$ is an even integer
$\Rightarrow q$ is an even integer
Thus p and q are both even which is a contradiction that both of p & q are not even. Hence $\sqrt{2}$ is an irrational number.

OR

Let $\sqrt{2}$ be a rational number and
$$\sqrt{2} = \frac{a}{b} \qquad \text{where } a, b \in \mathbb{Z},\ b \neq 0$$
$$\Rightarrow a^2 = 2b^2 \longrightarrow (1)$$
$\Rightarrow a^2$ is an integer

(31)

Theorem Prove that b/w two different rational numbers, there lie a rational no. Also prove that b/w any two different rational nos, there lie an infinite no of rational nos

OR

Prove that set of rational nos is dense.

Sol let a & b be two distinct rational nos with $a < b$

$$a < b$$

$$\Rightarrow a + a < a + b$$

$$a < \frac{a+b}{2} \rightarrow (1)$$

Also $a < b$

$$\Rightarrow a + b < 2b$$

$$\Rightarrow \frac{a+b}{2} < b \rightarrow (2)$$

$$\Rightarrow a < \frac{a+b}{2} < b$$

$\because$ $a, b, 2$ are rational, so is $\frac{a+b}{2} = r_1$

$\Rightarrow \exists$ a rational r_1 b/w a & b

Now a & r_1 are two rational no.

$\Rightarrow r_2 = \frac{a+r_1}{2}$ lies b/w a & r_1 i.e

$$a < r_2 < r_1$$

r_1 & b are two rational no

$r_3 = \frac{r_1+b}{2}$ lies b/w them i.e $r_1 < r_3 < b$

Thus $a < r_2 < r_1 < r_3 < b$

Continuing this process, there lie infinitely many rational nos between two rational nos

$\Rightarrow$ a is an even integer

Let $a = 2m$ $\quad m \in \mathbb{Z}$ $\quad$ where m may be even or odd

putting in ①

$$4m^2 = 2b^2$$

$$b^2 = 2m^2 \quad \longrightarrow ②$$

$\Rightarrow$ b^2 is an even integer

$\Rightarrow$ b is an even integer

Let $b = 2n$ $\quad$ where $n \in \mathbb{Z}$

putting it in ②

$$(2n)^2 = 2m^2$$

$$\Rightarrow m^2 = 2n^2$$

$\Rightarrow$ m^2 is an even integer

$\Rightarrow$ m is an even integer.

It is a contradiction because m is any integer even or odd according to assumption.

Thus $\sqrt{2}$ is an irrational number.

Question# Between Any two rational numbers, there lies another rational number. Also prove that b/w any two rational numbers there must (lie) be infinitely many rational numbers.

Sol# Let a & b any two rational numbers. Then their average $\frac{a+b}{2}$ is a rational number which lies between a & b. Thus between any two rational numbers their lies a ration number.

Now suppose that no of rational number between a & b is finite and they are

$$r_1, r_2, r_3 \text{ - - - - - - } r_n \text{ in all}$$

Then $\frac{r_1+r_2}{2}$ will be a rational number between r_1 & r_2 and hence between a & b. Also it will be surely different from $r_1, r_2 \text{ - - - - } r_n$. This condradicts the given situation. Hence number of rational numbers bet a & b is not finite.

Remarks Since there are infinitly many rational numbers between any two rational numbers, if we are given a certain rational number r, we can not speak of the "next largest" rational number.

Lamma# Rational numbers are not adjacent i.e. there no rational number closest or immediately adjacent to any rational number

Proof# Let x be any rational number. and let y is a rational no immedialtly adjacent to it. Then then the mid point $z = \frac{x+y}{2}$ is rational and would lie halway between x & y. This contradicts the property of y of being adjacent to x. Hence. There is no adjacent rational number like y.

Remarks from the above Lamma it comes out that there is no gap between adjacent rationals devoid (completly lacking) of other rationals, no gap to be "filled in" by irrationals. In fact there exist infinitely many of both types of numbers, whether rational or irrational. These sets of numbers, rather than interspersed in an alternating fashion, are thoroughly blended together (mixed together)

Proofs of Completeness Axiom of Real numbers.

The completeness axiom of R distingishes the real numbers from other ordered fields. This property can be stated in several equivalent ways, each more or less easy to believe. In a particular situation one form of completeness may seem easier to apply than other. What completeness says is that there are holes (gaps) in the real number system as there are in the rational number system. We prove this property by formal and informal methods as under

Informal Proof

Completeness axiom informally (roughly) means that real numbers are Complete in the sense that there are no "holes" in the real line.

Suppose that There is a hole in the real line which cuts the real line into two disjoint pecieces. Let S be the set all those points strictly to the left of the cut. Clearly S is not empty and S is bounded above by any real number to the right of the hole. Hence by Completeness axiom S has a least bound u in R. The point u must be the cut point i.e the cut could not have occurred a 'hole' In other words R has no holes.

Formal Proof

Consider the open interval (a subset of real no)

$S = (a \;\; b) = \{x \in R : a < x < b\}$

Then clearly $\quad x < b \qquad \forall x \in S$

$\Rightarrow b$ is an upper bound for S.

Also let $x \in S = (a, b)$ i.e $a < x < b$

Then $y = \frac{x+b}{2}$ is in S and larger than x & less than b

$\Rightarrow$ If we suppose that x belonging to S is an upper bound of S we can always find a number y of the type $y = \frac{x+b}{2}$ in S which is larger than x.

Hence x can not be upper bound.

But $x < b$ and b is an upper bound of S

$\Rightarrow$ no element ~~of~~ of R less than b can be an upper bound for S.

Thus $\sup S = b$ in R

Here (a, b) is a purely an arbitrary subset of R, which is bounded above and has the least upper bound in R. Therefore

R is a Complete set.

The Q. of Rational Numbers is not Complete

Informaly The set of rational nos, is not Complete in the sense there are holes in it. Which can be. informally proved as

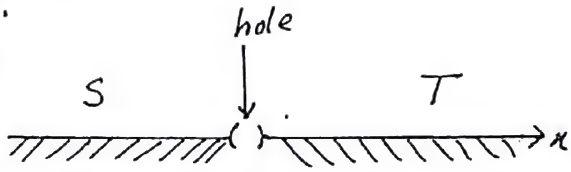
Consider two sets (subsets) of rational numbers as

$S = \{ x : x \in Q \text{ \& } x < \sqrt{2} \}$

$T = \{ x : x \in Q \text{ \& } x > \sqrt{2} \}$

Then S is the set of all rationals to the left of $\sqrt{2}$ and T is the set of rationals to the right of $\sqrt{2}$. Each rational number is either in S or in T.

The irrational number $\sqrt{2}$ (system has) is between S and T. It represents a hole in the rational number system.



The rational line has a hole at $\sqrt{2}$

Proof-II (Formal)

Here we explore some subsets of Q which has no lup (largest number) or glb (The smallest number) in Q.

Let $A = \{ p \in Q : p > 0, \ p^2 < 2 \}$

and $B = \{ p \in Q : p > 0, \ p^2 > 2 \}$

We prove that the set A has no greatest number in Q and. Set B has no smallest number in Q: In other words The set A has least upper bound and set B has no greatest lower bound.

More explicitly we are to show that for every p in A we can find a rational q in A such that $q > p$ and for every p in B we can find a rational q in B such that $q < p$.

We assovate with each rational $p > 0$ a rational number q as under.

$$q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2} \longrightarrow ①$$

$$= \frac{2(p+1)}{p+2}.$$

$$q^2-2 = \left[\frac{2(p+1)}{p+2}\right]^2 - 2.$$

$$= \frac{4(p^2+2p+1)}{(p+2)^2} - 2.$$

$$= \frac{4p^2+8p+4-2p^2-8p-8}{(p+2)^2}$$

$$= \frac{2p^2-4}{(p+2)^2} = \frac{2(p^2-2)}{(p+2)^2} \longrightarrow ②$$

For set A #

If $p \in A$, then $p^2-2<0$

and from ①

$$q = p + \text{Some +ve quantity}$$

$$\Rightarrow q > p$$

Also from ② $\quad q^2-2<0$

$$\Rightarrow q<2$$

Thus $q \in A$

$\Rightarrow$ For every p in A we can always design a rational no $q = p - \frac{p^2-2}{p+2}$ which is rational, greater than p and is in A. Hence 'A' has no least upper bound in Q. So least upper bound property fails to hold in Q and Q is not complete.

For Set B

If $p \in B$, then $p^2-2>0$ and from ①

$$q = p - \text{some quantity}.$$

$$\Rightarrow q<p$$

Also from ② $\quad q^2-2>0 \Rightarrow q^2>2.$

$$\Rightarrow q \in B$$

$\Rightarrow$ For every p in B we can always find a rational

$q = p - \frac{p^2+2}{p+2}$ which is rational, less than p and in B. Hence B has no greatest lower bound in Q. So glb property fails to hold in Q and Q is no complete.

OR

For set A

For every +ve rational number r satisfying $r^2 < 2$ i.e for +ve rational no r in A, we prove that we can find a larger rational no $r+h$ $(h>0)$ for which $(r+h)^2 < 2$ i.e $r+h \in A$

Let $h < 1$

Then $h^2 < h$.

Now $(r+h)^2 = r^2+2rh+h^2$

$< r^2+2rh+h \qquad \because h^2 < h$

$\Rightarrow (r+h)^2 < r^2+2rh+h$

Now find such value of h for which $r^2+2rh+h=2$

$r^2+2rh+h = 2.$

$(2r+1)h = 2-r^2$

$h = \frac{2-r^2}{2r+1}$

Hence for every r in A we can always find $h = \frac{2-r^2}{2r+1}$ such that $r+h$ is greater than r and is in A. Therefore A has no lub.

For set B We prove that for every +ve rational number s satisfying condition $s^2 > 2$ we can always a smaller rational number $s-k$ $(k>0)$ for which. $(s-k)^2 > 2$ i.e $s-k$ is in B.

We may assume $k > 1$

Then $k^2 > k$.

$(s-k)^2 = s^2-2sk+k^2 > s^2-2sk+k \quad (\because k^2 > k)$

setting $r^2-2rk+k = 2 \Rightarrow k = \frac{2-s^2}{1-2s}$

Hence for every r in B we can always find $k = \frac{2-s^2}{1-2s}$

such that $s-k$ is less than s and is in B, is rational.
Therefore B has no glb

OR

Consider a subset S of Q as
$$S = \{x : x \in Q, \ -\sqrt{2} \leq x \leq \sqrt{2}\}.$$
Then S is bounded above, say by 1.42 in Q.
Now clearly $\sqrt{2}$ is an upper bound of S
For $x \in [-\sqrt{2}, \sqrt{2}]$, $x \in Q$
$$y = \frac{x+\sqrt{2}}{2} \text{ is Larger than } x \text{ and less}$$
than $\sqrt{2}$. Thus if x is an upper bound, Then $\frac{x+\sqrt{2}}{2}$ greater than x is in $S = \{x \in R, \ -\sqrt{2} \leq x \leq \sqrt{2}\}$
$\Rightarrow$ $\sup S = \sqrt{2}$ does not exist in Q, where it does exist in R
Thus R is complete but Q is not complete.

Exercise # Show that $\sqrt{5}$ is not rational number

Sol # Let $\sqrt{5}$ be a rational number. Then
$$\sqrt{5} = p/q \quad p, q \in Z, \ q \neq 0$$
Let p & q are relatively prime numbers
squaring
$$5 = \frac{p^2}{q^2}$$
$$p^2 = 5q^2 \longrightarrow (1)$$
$\because$ $5q^2$ is a multiple of 5
$\therefore$ p^2 is a multiple of 5
$\Rightarrow$ p is a multiple of 5
Let $p = 5m$
putting in (1)
$$(5m)^2 = 5q^2$$
$$25m^2 = 5q^2$$
$$q^2 = 5m^2$$
$\Rightarrow$ q is also a multiple of 5

Since p and q are multiple of 5, so
p/q is not in the lowest form.
Hence $\sqrt{5}$ is not a rational number.

Theorem # If n is an integer which is not a perfect, then $\sqrt{n}$ is irrational.

Proof # Case-I #

When n contains no square factor greater than 1

Let $\sqrt{n}$ be rational and

$$\sqrt{n} = \frac{a}{b} \qquad a, b \in \mathbb{Z} \ \& \ (a,b)=1$$

$$\Rightarrow a^2 = n b^2 \longrightarrow (1)$$

$$\Rightarrow n / a^2 \Rightarrow n / a \longrightarrow (2)$$

$$\Rightarrow a = cn \quad \text{where } c \in \mathbb{Z}$$

putting in (1)

$$(cn)^2 = nb^2$$

$$\Rightarrow c^2n^2 = nb^2$$

$$\Rightarrow b^2 = c^2 n$$

$$\Rightarrow n/b^2 \Rightarrow n/b$$

Thus a & b are both multiple of n
This gives a contradiction to our assumption that $(a,b)=1$
Hence our assumption is wrong and $\sqrt{n}$ is irrational

Case-II #

When n has a square factor

Let $n = m^2 k$ where $k > 1$ and k has no square factor greater than 1

$\sqrt{32} = \sqrt{2 \cdot 16}$
16 is a square factor

Then $\sqrt{n} = m\sqrt{k}$
Then obviously $\sqrt{k}$ is an irrational number as proved above & m is rational
Thus $\sqrt{n}$ being the product of a rational & an irrational is irrational (Proved)

Lamma # (a) # The sum of a rational and an irrational number is an irrational number

(b) #. The product of a rational and an irrational number is an irrational number.

Proof # a # Let r be an rational & s an irrational no

Let $t = r + s$ and let t be rational

Then $s = t - r$ is the difference of two rational numbers and so is a rational. This a contradiction. Hence $r + s$ is an irrational number.

OR

Let r be a rational number and s be an irrational number. Then

$\because$ r is rational

$\therefore$ $r = m/n$ where $m, n \in Z$, $n \neq 0$

Let $r + s$ be a rational number and

$r + s = p/q$ $\quad p, q \in Z$ $\quad q \neq 0$

$\Rightarrow \frac{m}{n} + s = p/q$

$$s = p/q - m/n = \frac{pn - mq}{nq} = \frac{l}{t}$$

where l & t are Integers and $t = nq \neq 0$

Therefor s is a rational number which is a contradiction to that s is a an irrational number.

Note Similarly it can be proved that difference of a rational and an irrational number is an irrational no.

(b) # Let r be ration and s be an irrational number. Let rs be a rational number &

$rs = p/q$ $\quad p, q \in Z$; $q \neq 0$

$\Rightarrow s = \frac{p}{rq}$ $\quad p \in Z$, $rq \in Z$

$\Rightarrow$ s is a rational which is a contradiction. Hence rs is irrational

Exercise# Prove that $\sqrt{12}$ is irrational number. i.e

Sol $\sqrt{12} = 2\sqrt{3}$ where $\sqrt{3}$ is an irrational no

$\Rightarrow \sqrt{12}$ being a product of an irrational and rational numbers is irrational.

OR

Let $\sqrt{12}$ be rational &

$\sqrt{12} = p/q$ $(p,q)=1$ p & q has. no Common factor

$\Rightarrow p^2 = 12q^2 \rightarrow (a)$

$\Rightarrow p^2 = 2^2 \cdot 3 q^2$

$\Rightarrow 2\cdot 3 / p^2$ & $3/p^2$

$\Rightarrow 6/p^2$ & $3/p \rightarrow ①$

Now $6/p^2$

$\Rightarrow 6/p \Rightarrow p = 6p_1$ $p_1 \in \mathbb{Z}$

Putting in (a)

$36p_1^2 = 12q^2$

$\Rightarrow q^2 = 3p_1^2$

$\Rightarrow 3/q^2 \Rightarrow 3/q$

Thus 3 is a common factor of p & q

which contradicts our assumption that $(p,q)=1$

Hence $\sqrt{12}$ is an irrational number.

Exercise# 1)# Prove that $\sqrt{3}+\sqrt{2}$ is an irrational no

2)# Find all rational values of x at which $y = \sqrt{x^2+x+3}$ is a rational number.

Sol# 1)# $\because \sqrt{3}+\sqrt{2}$ is sum of two irrational numbers

$\therefore \sqrt{3}+\sqrt{2}$ is an irrational number

OR

Assume on the contrary that $\sqrt{3}+\sqrt{2}$ is rational

Then $\sqrt{3}-\sqrt{2} = \dfrac{1}{\sqrt{3}+\sqrt{2}}$ is rational since it is quotient of two rationals

Now $\sqrt{2} = \frac{1}{2}[\sqrt{3}+\sqrt{2}-(\sqrt{3}-\sqrt{2})]$ is rational which contradicts the irrational nature of the number $\sqrt{2}$. Hence, the supposition is wrong and the number $\sqrt{3}+\sqrt{2}$ is irrational.

(b) Let x & $y = \sqrt{x^2+x+3}$ are rational numbers.

Then $y - x = q$ is also rational.

Now $y - x = \sqrt{x^2+x+3} - x = q$

$$\Rightarrow \sqrt{x^2+x+3} = q+x$$

$$\Rightarrow x^2+x+3 = (q+x)^2$$
$$= q^2+x^2+2xq$$

$$\Rightarrow x = \frac{q^2-3}{1-2q}$$

Here $1-2q \neq 0 \Rightarrow q \neq 1/2$

Now we prove the reverse that y is rational when $x = \frac{q^2-3}{1-2q}$

$$y = \sqrt{x^2+x+3}$$
$$= \sqrt{\frac{(q^2-3)^2}{(1-2q)^2} + \frac{q^2-3}{1-2q} + 3}$$

$$= \sqrt{\frac{q^4-2q^3+7q^2-6q+9}{(1-2q)^2}}$$

$$= \frac{\sqrt{(q^2-q+3)^2}}{\sqrt{(1-2q)^2}}$$

$$= \frac{(q^2-q+3)}{|1-2q|} \qquad (q \neq 1/2)$$

This expression is rational at any rational q not equal to $1/2$

Theorem # Let A & B be two bounded sets of real numbers with

$$a = \operatorname{Sup} A \quad , \quad b = \operatorname{Sup} B.$$

Let C denote the set

$$C = \{x+y \mid x \in A, y \in B\}$$

Then $a+b = \operatorname{Sup} C$

Proof # Let z be any element of C.

Then by definition of C

$$z = x+y \qquad x \in A \,.\, y \in B$$

$\because$ a & b are sups of A & B

$\therefore$ $x \leq a$ & $y \leq b$

$\Rightarrow$ $x+y \leq a+b$

$\Rightarrow$ $z \leq a+b$

$\Rightarrow$ $a+b$ is an upper bound of $A+B=C$

Let c be any upper bound of C. We must show that $a+b \leq c$

$\because$ a & b are sups of A & B

$\therefore$ For every +ve number ϵ $\exists$ exists numbers x in A & y in B such that

$$a-\epsilon < x \quad , \quad b-\epsilon < y$$

Adding

$$a+b-2\epsilon < x+y \leq c$$

$\Rightarrow$ $a+b-2\epsilon < c$

$\Rightarrow$ $a+b \leq c+2\epsilon$

But since ϵ is arbitray, therefore

$$a+b < c$$

Lamma # Let n be a positive integer. No +ve integer m satisfies the Inequality

$$n < m < n+1$$

i.e There is no integer b/w two Consecutive +ve integers.

Proof# Suppose $\exists$ a +ve integer m such that

$$n < m < n+1$$

Then from inequality $n < m$, we have that

$$m-n \text{ is a +ve integer}$$

and from inequality $m < n+1$, we have

$$m-n < 1$$

Thus $m-n$ is a +ve integer less than 1 which is not possible. Hence the result.

Theorem# (Well-ordering theorem)

If X is a non void subset of the +ve integers, then X contains a least element i.e $\exists$ an $a \in X$ such that $a \leq x \quad \forall x \in X$

Proof# We use induction on n.

Let $S(n)$ be a statement "If $n \in X$, then X contains a least element"

If $1 \in X$, then 1 is the least element of X because if n is any +ve integer, then $n \geq 1$

Assume that $S(k)$ is true i.e

If $k \in X$, then X contains a least element.

Suppose $k+1 \in X$

$\because$ $S(k)$ is true

$\therefore$ $X \cup \{k\}$ contains a least element m.

If $m \in X$, then m is the least element of X

If $m \notin X$, $m = k$ and $k \leq x \; \forall x \in X$

Since $k \notin X$, we have $k+1 \leq x \quad \forall x \in X$

In this case $k+1$ is the least element of X

Thus $S(k+1)$ is true and hence the result by induction

Theorem# The set of +ve integers, P is not bounded above.

P.T.O

Proof Let P be bounded, then by least upper bound axiom P has a least upper bound a.
$\because$ $a-1$ is not an upper bound of P
$\therefore$ $\exists$ $n \in P$ such that

$$a-1 < n$$
$$\Rightarrow a < n+1$$

$\Rightarrow a$ is not an upper bound of P

which is a contradiction. Thus the result.

Corollary # 1: The set of real numbers is Archimedian ordered i.e if a & b are +ve real numbers, $\exists$ a +ve integer n such that

$$a < nb$$

Proof # Since P is not bounded, therefore we can seek always a +ve integer n such that

$$\frac{a}{b} < n$$
$$\Rightarrow a < nb$$

Corollary # 2: If ϵ is a +ve real number, there exists a +ve integer N such that $\frac{1}{N} < \epsilon$

Proof # Let $a = 1$, $b = \epsilon$, then 1 & ϵ are +ve real numbers. By Archimedian's Principle $\exists$ $N \in P$ s.t

$$1 < N\epsilon$$
$$\Rightarrow \frac{1}{N} < \epsilon$$

Archimedian Principle

If $x \in R$, $y \in R$ and $x > 0$, then is a +ve integer n such that

$$nx > y$$

Proof # Let the proposition is not true for any +ve integer n i.e for any +ve integer n we have.

$$nx \leq y$$

Let $A = \{nx : n \in N\}$

$\because \quad nx \leq y \quad \forall n \in N$

$\therefore$ y is an upper bound for A and A is bounded above.

By least upper bound property of R. A has a least upper bound in R

let $\sup A = \alpha$

$\because \quad x > 0$

$\therefore \quad \alpha - x$ is not an upper bound of A

$\Rightarrow \quad \alpha - x < mx$ for some some m in N

$\Rightarrow \quad \alpha < (m+1)x \in A$

which is impossible because α is an upper bound for A

Hence $nx > y$

Remarks If we take $x = 1$, $y \in R$, then $\exists\, n \in N$ s.t

$$1 \cdot n > y$$

or $\quad y < n$

Now if we take real number $x = 1$ as +ve real number, then for any real number y $\exists$ an $n \in N$ such that

$$y < n$$

This is another statement of Archmedian Principle. and can be proved as

Theorem # (Archimedian Principle)

For every real number x, there is an integer s.t $x < n$

Proof# Suppose there exists no integer satisfying $x < n$

But $n \leq x \quad \forall n \in \mathbb{Z}$

Then $\mathbb{Z} = \{n : n \leq x\} \neq \phi$ is bounded above by x

By lub property of real numbers there is a real numbers b such that $b = \sup \mathbb{Z}$

Obviousely if $n \leq x$, then $n+1 \leq x \quad \forall n \in \mathbb{Z}$

$\Rightarrow \quad n+1 \leq b \quad (\because b$ is an upper bound for $\mathbb{Z})$

$\Rightarrow \quad n \leq b-1 \quad \forall n \in \mathbb{Z}$

which is contrary to the fact that $b = \sup \mathbb{Z}$.

Hence there is an integer n such that $x < n$

OR

Let S be the set of (real) integers k such that $k \leq x$ i.e

$$S = \{k \in \mathbb{Z},\ k \leq x\}$$

Then S is bounded above by x

By lub property of real numbers S has a least upper bound y in R

$\because \quad y$ is lub for S

$\therefore \quad y - \frac{1}{2}$ can not be an upper bound for S

$\Rightarrow \exists$ a $k \in S$ such that

$$k > y - \frac{1}{2}$$

$\Rightarrow k+1 > y - \frac{1}{2} + 1$

$\Rightarrow k+1 > y + \frac{1}{2} > y$

$\Rightarrow k+1 > y > k \qquad \because k \in S$ & y is lub of S

~~But~~ $\Rightarrow k+1 \notin S$

$\Rightarrow k+1 \nleq x$

But $k+1 > x$

Hence proved

Theorem # If $x \in R, y \in R$ and $x < y$, Then there exists a $p \in Q$ such that
$$x < p < y$$
OR

Between any two real numbers there lies a rational number.

Proof Given that $x < y$
$$\Rightarrow \quad y - x > 0$$
By Archimedes' principle $\exists$ a +ve integer n such that
$$1 < n(y-x) = ny - nx$$
$$\Rightarrow \quad nx + 1 < ny \quad \longrightarrow (1)$$
Again $1 \in R$, $nx \in R$ and $1 > 0$, so there exists a +ve integer m_1 such that
$$nx < m_1 \quad \longrightarrow (2)$$
Also since $1 \in R$, $1 > 0$, $-nx \in R$, $\exists$ an integer m_2 such that
$$1 \cdot m_2 > -nx$$
$$\Rightarrow -m_2 < nx \quad \longrightarrow (3)$$
By (2) & (3)
$$-m_2 < nx < m_1$$
Thus there is an integer m (with $-m_2 \leq m \leq m_1$) s.t
$$m-1 < nx < m$$
$$\Rightarrow m < 1+nx \quad ; \quad nx < m$$
$$\Rightarrow nx < m < 1+nx < ny \qquad \text{by (1)}$$
$$\Rightarrow nx < m < ny$$
$$\Rightarrow x < \frac{m}{n} < y$$
where $m/n = p$ is the rational number

OR

Case-I First we consider the case that $y > 0$.
Then we may choose a +ve integer n s.t.

$$y - x > \frac{1}{n}$$

Let m be the least +ve integer such that

$$y \leq m/n$$

If $m = 1$, clearly $\frac{m-1}{n} < y$.
If $m > 1$, $m-1$ is a +ve integer less than m
and Thus

$$\frac{m-1}{n} < y \longrightarrow (1)$$

Now $y - x > \frac{1}{n}$

$$\Rightarrow -(x-y) > 1/n$$

$$\Rightarrow x - y < -\frac{1}{n}$$

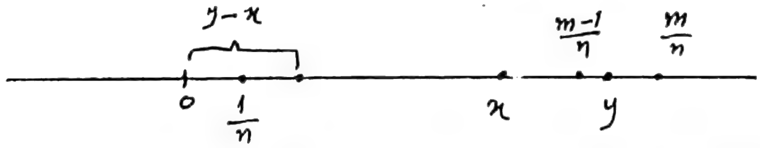
$$x = y + (x-y) < \frac{m}{n} + \left(-\frac{1}{n}\right)$$ $\because y \leq \frac{m}{n}$, $x - y < -\frac{1}{n}$

$$\Rightarrow x < \frac{m-1}{n} \longrightarrow (2)$$

By (1) & (2)

$$x < \frac{m-1}{n} < y$$

Hence $p = \frac{m-1}{n}$ is the rational



Case-II Suppose that x & y are arbitrary real numbers with $x < y$. We have established the theorem in case of $y > 0$

Choose a +ve integer n such that $y + n > 0$ (We may choose always such integer because set of +ve integer is not bounded above and we may select larger and large +ve integer which will fulfil our condition.)

Now shifting x to $x+n$, we have two real numbers $x+n$, $y+n$ such that $y+n > 0$ and by case I. There exists a rational no r s.t

$$x + n < r < y + n$$

$$\Rightarrow x < r - n < y$$ $r - n = p$ is rational no

Theorem
For any real number x, there exists a unique integer m such that

$$\cancel{m \leq x < m+1}$$

Proof Let $A = \{n : n \in Z,\ n \leq x\}$.
Then $A \neq \phi$ and A is bounded above by x

$\therefore$ ~~By Completeness property~~. A has $\boxed{\text{lub say } m}$ greatest integ m such that $m \leq x$

$\therefore$ m is the greatest integer satisfying $m \leq x$

$\therefore$ $x < m+1$

Hence $m \leq x < m+1$

Theorem If $x \in R$, $y \in R$ and $x < y$ then $\exists$ a $p \in Q$ such that
$$x < p < y$$
OR
between any two real numbers there lies a rational number

Proof let two real no be
x & y & $x < y$
$\Rightarrow y - x > 0$
~~By Archemedian~~ property in R $\exists$ $n \in N$
s. that
$n(y-x) > 1$
$\Rightarrow ny > nx+1 \longrightarrow (1)$
$nx+1 < ny \longrightarrow (2)$
Also $\exists$ a unique integer m such that
$m-1 \leq nx < m$
$m \leq nx+1 < m+1$
$ny > nx+1 \geq m > nx$
$nx < m < ny$
$x < \frac{m}{n} < y$
$\frac{m}{n} \in Q$ is a ration no between x & y

Theorem# If x & y are two real numbers with $x < y$, then is an irratational number s such that

$$x < s < y.$$

OR

Between two real numbers there is an irrational numbers.

Proof# Let x & y be real numbers such that

$x < y$

Then $\frac{x}{\sqrt{2}}$ & $\frac{y}{\sqrt{2}}$ also real numbers.

Now between two real real numbers $\exists$ a rational number.

Therefore $\exists$ a rational number r such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$\Rightarrow x < \sqrt{2}r < y$$

Here $\sqrt{2}r$ is an irrational number.

OR

$x-\sqrt{2}$ & $y-\sqrt{2}$ are real numbers such that

$$x-\sqrt{2} < y-\sqrt{2}$$

$\exists$ a rational number r such that

$$x-\sqrt{2} < r < y-\sqrt{2}$$

$$\Rightarrow x < r+\sqrt{2} < y$$

where $r+\sqrt{2}$ is an irrational number.

More General Proof

Let x & y be any two real numbers such that

$x < y$.

Case-I: Let α be any +ve irrational number.

Then x/α, y/α are real number &

$$\frac{x}{\alpha} < y/\alpha$$

Since b/w two real numbers $\exists$ a rational number, therefo

there is a rational number r such that

$$\frac{x}{\alpha} < r < y/\alpha$$

$$\Rightarrow x < \alpha r < y$$

where αr is an irrational no b/w x & y

Case-II Let β be a -ve irrational number

Then $\frac{x}{\beta}$ & $\frac{y}{\beta}$ are real numbers and

$$\frac{x}{\beta} > \frac{y}{\beta}$$

$\Rightarrow \exists$ a rational number r such that

$$y/\beta < r < x/\beta$$

$$\Rightarrow y > \beta r > x \qquad \because \beta < 0$$

$$\text{or} \quad x < \beta r < y$$

where βr is an irrational number between x & y

Theorem #(a): For any real number x, there is an integer n such that

$$n \leq x < n+1.$$

(b) # For every real number x, there is a set A of rational numbers such that

$$x = \operatorname{Sup} A$$

i.e every real number x is sup of some set of rational numbers or Every set of rational number has supremum in R but not in Q.

Proof # (a) Let x be any real number.

Then by Archimedes' Principle there is an integer m such that $x < m$.

Let $A = \{m \in \mathbb{Z} : m \leq x\}$

Then A is bounded above by x and has the greatest integer n such that $n \leq x$

$\because$ n is the greatest integer satisfying $n \leq x$

$\therefore$ $x < n+1$

Hence $n \leq x < n+1$ Proved.

(b) #

Let A be the set of all rational numbers less than x i.e

$$A = \{ q \in Q : q < x \}$$

Then A is bounded above by x.

By least upper bound property of real numbers, there is a real number α in R such that

$$\alpha = \operatorname{Sup} A$$

Clearly $\alpha \leq x$

If $\alpha = x$, then theorem is proved.

If $\alpha < x$, then we can find a rational number r such that

$$\alpha < r < x$$

$\because r < x$ $\therefore r \in A$

Thus α is less than one of elements of A which is impossible because α is supremum of all rational numbers less than x.

Hence $\alpha = x$.

So there is a subset A of rational numbers such that

$$x = \operatorname{Sup} E.$$

Problem # For any +ve integer for every j in N

$$2^{j-1} \leq j!$$

Sol It can easily be proved by induction.

Exercise # Prove that the set $S = \{ x_k = \sum_{j=0}^{k} \frac{1}{j!} : k \in N \}$

Sol # $2^{j-1} \leq j!$, $\frac{1}{2^{j-1}} \geq \frac{1}{j!}$

$$x_k = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}$$

$$\leq 1 + \frac{1}{2^0} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}$$

$$= 1 + \frac{1-(1/2)^k}{1-(1/2)} = 1 + 2\left[1 - \left(\tfrac{1}{2}\right)^k\right] < 3$$

Consequently, S is bounded above by 3.

S is bounded below by 1 and is therefore contained in the closed interval $[1\ 3]$. It is easy to see that $\inf S = 1$, but the value of $\sup S$ is not at all apparent. In fact, $\sup S$ is the number e, an irrational number whose approximate value is 2.71828

Exercise # $S = \{ (1 + 1/k)^k : k \text{ in } N\}$. Prove that S is bounded above.

Sol
$$\left(1+\frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \frac{k(k-1)}{2!}\left(\frac{1}{k}\right)^2 + \frac{k(k-1)(k-2)}{3!}\left(\frac{1}{k}\right)^3 + \cdots \frac{1}{k^k}$$
$$\le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}$$
$$\le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}$$
$$= 1 + \frac{1-(1/2)^k}{1-1/2}$$
$$= 1 + 2\left[1 - \left(\frac{1}{2}\right)^k\right] < 3 \qquad \forall k \text{ in } N$$

Thus the set S is bounded above by 3

Bernoulli's Inequality

$$(1+x)^n \geq 1 + nx \qquad \forall x > -1$$

Exercise # There is a unique real number $x > 0$ such that $x^2 = 2$

Sol

* Existence #

Let $A = \{q \in Q : q^2 \le 2\}$

Then A is non-empty because $1 \in A$

If $x \in A$, $x > 1$, then $x < x^2 \le 2$.

$\Rightarrow$ A is bounded above by 2.

By lub property of real numbers $\exists$ a real number α such that $\alpha = \sup A$

Then α does not belong to Q because for every

rational no r satisfying $r^2 < 2$ we can alway find a larger rational $r+h$ such that $(r+h)^2 < 2$ as
let $h < 1$. Then $h^2 < h$
and $(r+h)^2 = r^2 + 2rh + h^2 < r^2 + 2rh + h$.
Here if we let $r^2 + 2rh + h = 2$, then

$$h = \frac{2-r^2}{2r+1}$$

Now for every rational r we can always find h from the formula $h = \frac{2-r^2}{2r+1}$ such that

$$(r+h)^2 < 2$$

Similarly α^2 is not greater than 2 because for every real number greater than 2 we can find a greater real number. Hence

$$\alpha^2 = 2$$

Uniqueness

Let x_1 & x_2 be two real numbers such that $x_1^2 = 2$ & $x_2^2 = 2$
Then

$$x_1 - x_2 = \frac{(x_1-x_2)(x_1+x_2)}{x_1+x_2}$$

$$= \frac{x_1^2 - x_2^2}{x_1+x_2}$$

$$= 0$$

$$\Rightarrow x_1 = x_2$$

Hence uniqueness is proved.

Theorem # For every real $x \geq 0$ and for every integer $n > 0$, there is one and only one real y such that $y^n = x$
This no. y is written as $\sqrt[n]{x} = x^{1/n}$

Proof #

Uniqueness

Let $x^{1/n} = y_1$ & $x^{1/n} = y_2$

and $y_1 < y_2$
$\Rightarrow y_1^n = x = y_2^n$
It is impossible because $y_1 < y_2$ ...

∴ Hence there is only one real value of y

Now we shall prove that $y^n = x$.

to prove $y^n = x$ we will show that each of inequalities $y^n < x$ & $y^n > x$ leads to Contradiction.

Let $A = \{t \in R : t \geq 0, t^n < x\}$

If $t = \frac{x}{1+x}$, then $0 < t < 1$

Hence $t^n < t < x$

$\Rightarrow t \in A$ and A is non-empty.

If $t > 1+x$, then $t^n > t > x$

$\Rightarrow t \notin A$

$\Rightarrow 1+x$ is an upper bound of A.

OR

It can be proved as

Suppose there exists $t \in A$ such that $t > x+1$.

Then

$$x \geq t^n > (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \geq nx$$

$\Rightarrow x > nx$ which is impossible.

Hence all elements of A are less or equal to $x+1$ and $x+1$ is an upper bound of A.

By the least-upper bound axiom, A has a least upper bound.

Let $\operatorname{Sup} A = y$

we shall show that $y^n < x$ & $y^n > x$ gives Contradiction

Then it will be obvious that $y^n = x$

Case-I # $y^n < x$.

The identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \dots + a^{n-1})$

yields the inequality

$b^n - a^n < (b-a)n\, b^{n-1}$ where $0 < a < b$ → (A)

choose h so that $0 < h < 1$ &

$$h < \frac{x-y^n}{n(y+1)^{n-1}}$$

Let $a=y$, $b=y+h$ in Ⓐ , Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x-y^n$$

$$\Rightarrow (y+h)^n < x$$

$\Rightarrow y+h \in A$ which is a contradiction y is supremum of A

Hence $y^n \not< x$

Case - II Let $y^n > x$ and

$$k = \frac{y^n - x}{ny^{n-1}} = \frac{y^n}{ny^{n-1}} - \frac{x}{ny^{n-1}} = \frac{y}{n} - \frac{x}{ny^{n-1}}$$

Then $0 < k < y$. If $t \geq y-k$, we conclude That

$$y^n - t^n = y^n - (y-k)^n < kny^{n-1} = y^n - x \qquad y^n - [y^n + kny^{n-1}]$$

Thus $t^n > x$ & $t \notin A$

$\Rightarrow y-k$ is an upper bound of A

But $y-k < y$ which contradicts the fact that y is the least upper bound of A

Hence $y^n = x$ (Proved).

Corollary # If a and b are +ve real numbers and n is a +ve integer, Then

$$(ab)^{1/n} = a^{1/n} b^{1/n}$$

Proof # Let $\alpha = a^{1/n}$, $\beta = b^{1/n}$

$$\alpha^n = a \qquad \beta^n = b$$

$$ab = \alpha^n \beta^n = (\alpha\beta)^n$$

$$\Rightarrow (\alpha\beta) = (ab)^{1/n}$$

$$a^{1/n} \cdot b^{1/n} = (ab)^{1/n}$$

The Extended Real Number System

The extended real number system consists of the real field R and two symbols $+\infty$ & $-\infty$. The original order in R is preserved and.

$$-\infty < x < +\infty \qquad \forall x \in R.$$

The extended Real number system does not form a field but following conventions are made.

(a) If x is real, then

$$x + \infty = +\infty \qquad x - \infty = -\infty$$

$$\frac{x}{\infty} = \frac{x}{-\infty} = 0$$

(b) $x \cdot (+\infty) = \infty \quad x\cdot(-\infty) = -\infty$ if $x > 0$

(c) $x \cdot (+\infty) = -\infty \,,\; x \cdot (-\infty) = +\infty$ If $x < 0$

$$\infty + \infty = \infty \qquad -\infty - \infty = -\infty$$

$$\infty \cdot (\pm\infty) = \pm\infty \qquad -\infty \cdot (\pm\infty) = \mp\infty$$

Absolute Value

Define a function $|\cdot|$ on R as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$|x|$ is called absolute value or modulus of a real number x. Geometrically $|x|$ represents the distance from x to origin 0 on real line.

Theorem # If x is a real number, then

(a) # $|x| \geq 0$

(b) # $|-x| = |x|$

(c) # $x \leq |x|$ & $-x \leq |x|$

Proof # (a) # By definition.

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

Hence in all cases $|x| \geq 0$

(b) # $|-x| = |x|$

Case I When $x > 0$, $-x < 0$

Hence $|-x| = -(-x) = x$

$|x| = x \quad \because x > 0$

$\Rightarrow |-x| = |x|$

Case-II When $x < 0$, $-x > 0$

Hence $|-x| = -x$

$\therefore |x| = -x \quad \because x < 0$

$\Rightarrow |-x| = |x|$

Then $|-x| = |x|$

Theorem

(a) # Let $\epsilon > 0$, then $|x| < \epsilon$ iff $-\epsilon < x < \epsilon$ and $|x| \leq \epsilon$ iff $-\epsilon \leq x < \epsilon$.

More generally $|x-a| < \epsilon$ iff $a-\epsilon < x < a+\epsilon$

(b) # $x \leq |x|$ & $-x \leq |x|$

(c) # $|x+y| \leq |x| + |y|$

(d) # $||x| - |y|| \leq |x-y|$

(e) # $|xy| = |x||y|$

(f) # $|x-z| \leq |x-y| + |y-z|$

Proof # $\because$ By definition $|x|$, $-|x| \leq x < |x|$

$\therefore$ if $|x| < \epsilon$

Then $-\epsilon < -|x| < x < |x| < \epsilon$

$\Rightarrow -\epsilon < x < \epsilon$.

OR

Let $|x| < \epsilon$

$\Rightarrow x < \epsilon$ & $-x < \epsilon$ [illegible]

Conversely let $-\epsilon < x < \epsilon$

Then if $x > 0$, we have

$$|x| = x \leq \epsilon$$

$$\Rightarrow |x| < \epsilon$$

If $x < 0$, we have $|x| = -x < \epsilon$

Thus in either case

$$|x| < \epsilon$$

(ii) If $x \geq 0$, then

$$x = |x| \leq |x|$$

$$\Rightarrow x \leq |x|$$

If $x < 0$, then

$$x = -|x| < |x|$$

Thus $x \leq |x|$

Again if $x \geq 0$, $-x \leq 0$

$$-x \leq x \leq |x|$$

$$\Rightarrow -x \leq |x|$$

If $x < 0$, $-x > 0$

$$|x| = -x \qquad \because x < 0$$

Now $\boxed{\cancel{-x / \leq / -x}}$ $\qquad \because -x > 0$

Let $y = -x > 0$

Then $y \leq |y| = |-x| = |x|$

$$\Rightarrow -x \leq |x|$$

(C) # $|x+y| \leq |x| + |y|$

Since $x \leq |x|$

$y \leq |y|$.

Adding two inequalities

$$x+y \leq |x| + |y| \longrightarrow ①$$

Also $-x \leq |x|$

$-y \leq |y|$

$$\Rightarrow -(x+y) \leq |x| + |y| \longrightarrow ②$$

Combining ① & ②

$$|x+y| \leq |x| + |y|$$

Note#. If x & y differ in sign, then $|x+y|$ is less than $|x|+|y|$. In all other cases $|x+y|$ equals to $|x|+|y|$

(e)# $|xy| = |x||y|$

Without the loss of generality, let $x \neq 0, y \neq 0$

We have $|x|^2 = x^2 \quad \forall x \in \mathbb{R}$.

$$\Rightarrow |xy|^2 = x^2y^2 = |x|^2|y|^2$$

$$= (|x||y|)^2$$

$$\Rightarrow 0 = |xy|^2 - (|x||y|)^2$$

$$= (|xy| - |x||y|)(|xy| + |x||y|)$$

$$\Rightarrow |xy| - |x|\cdot|y| = 0 \quad \because |xy| + |x||y| \neq 0$$

$$\Rightarrow |xy| = |x||y|$$

If any one of x & y is zero, then

$$|xy| = |0| = 0$$

$$|x||y| = 0|y| = 0 \qquad \text{let } x = 0$$

$$\Rightarrow |xy| = |x||y|$$

Thus $|xy| = |x||y|$

(d)# $x = (x-y) + y$

$$|x| = |(x-y) + y| \leq |x-y| + |y|$$

$$\Rightarrow |x| - |y| \leq |x-y| \longrightarrow (1)$$

Again

$$|y| = |x + (y-x)| \leq |x| + |y-x|$$

$$|y| - |x| \leq |y-x| = |x-y|$$

$$-(|x| - |y|) \leq |x-y| \longrightarrow (2)$$

By (1) & (2)

$$\big||x| - |y|\big| \leq |x-y| \quad \text{(Proved)}$$

(f) # $|x-z| = |(x-y)+(y-z)|$

$|x-z| \le |x-y| + |y-z|$ proved.

Lemma # Let x be a real number and $|x| < \epsilon$ for every +ve real number ϵ, however small it may be, then $x=0$

Proof # Let $x \neq 0$

$\because |x| < \epsilon$ for any $\epsilon > 0$

Let $\epsilon = \frac{|x|}{2}$

Then $|x| < \frac{|x|}{2}$ which is impossible

Hence our supposition is wrong and $x=0$

More generally let x be a real number and α a fixed real number such that

$|x| < \alpha\epsilon$ for any +ve $\epsilon \in R$

Then $x = 0$

Let $x \neq 0$

$\because |x| < \alpha\epsilon$ for any $\epsilon > 0$

Let $\epsilon = \frac{|x|}{2\alpha}$

Then $|x| < \alpha\epsilon = \alpha \cdot \frac{|x|}{2\alpha}$

$\Rightarrow |x| < \frac{|x|}{2}$

which is impossible. Hence our supposition is wrong and $x=0$

Theorem # A subset A of real number R is bounded iff there is a number m such that

$$|x| < m \quad \forall x \in A$$

Proof # Let A be any subset of real number. Then A is bounded if there exist two real numbers a & b such that

$$a < x < b \qquad \forall x \in A$$

This clearly gives a real number m such that

$$-m < a < x < m \qquad \forall x \in A$$

$$\Rightarrow \quad -m < x < m \qquad \forall x \in A$$

$$\Rightarrow \quad |x| < m \qquad \forall x \in A$$

Alternatively, A is bounded iff $\exists$ a bounded interval $]-m \quad m[$ which contains A.

in fact

If $m = \operatorname{Inf} A \qquad M = \operatorname{Sup} A$

Then

$$A \subseteq \,]m-\epsilon \quad M+\epsilon[\,, \quad \epsilon > 0$$

Euclidean Spaces

Euclidean n-space R^n is the cartesian product of n-copies of R i.e.

$$R^n = R \times R \times R \times \cdots \times R$$

It consists of all n-tuples $(x_1, x_2, \ldots x_n) = \underline{x}$, where $x_i \in R \quad \forall i$ i.e

$$R^n = \{\underline{x} : \underline{x} = (x_1, x_2, x_3 \ldots x_n), x_i \in R, 1 \le i \le n\}$$

R^n is a vector space over the real field with addition and scalar multiplication defined as under.

Addition#

Let $\underline{x} = (x_1, x_2, \dots x_n)$
$\underline{y} = (y_1, y_2 \dots y_n) \in R^n$

$$\underline{x} + \underline{y} = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in R^n$$

Scalar Multiplication#

For $\underline{x} = (x_1, x_2, x_3, \dots x_n) \in R^n$
$\alpha \underline{x} \in R^n \quad \forall \alpha \in R$ and.

$$\alpha \underline{x} = (\alpha x_1, \alpha x_2, \dots \alpha x_n)$$

The additive identity in R^n is zero vector
$\underline{O} = (0, 0, 0 \dots, 0)$
and it has the property
$\underline{x} + \underline{O} = \underline{O} + \underline{x} = \underline{x}$
The additive inverse of the vector $\underline{x} = (x_1, x_2 \dots x_n)$ is $-\underline{x} = (-x_1, -x_2, \dots -x_n)$ and it has the property

$$\underline{x} + (-\underline{x}) = \underline{O} = (-\underline{x}) + (\underline{x})$$

Addition is associative in R^n i.e

$$(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$$

Addition is Commutative in R^n i.e

$$\underline{x} + \underline{y} = \underline{y} + \underline{x}$$

Scalars multiplication has the following properties.
For all $\underline{x}, \underline{y}$ in R^n and c, d in R

i) $c(\underline{x} + \underline{y}) = c\underline{x} + c\underline{y}$
ii) $(c+d)\underline{x} = c\underline{x} + d\underline{x}$
iii) $c(d\underline{x}) = (cd)\underline{x}$
(IV) $0\underline{x} = \underline{O}$, $1 \cdot \underline{x} = \underline{x}$ and $(-1)\underline{x} = -\underline{x}$.

Unit Co-ordinate Vectors in R^n

The vectors

$\underline{u}_1 = (1, 0, 0, \cdots, 0)$, $u_2 = (0, 1, 0, \cdots, 0)$

$u_3 = (0, 0, 1, 0, 0, \cdots, 0)$, $\cdots$ $u_k = (0, 0, 0, \cdots 1, 0, 0 \cdots 0)$

$\cdots$ $u_n = (0, 0, 0, \cdots, 1)$ where 1 is at kth Co-ordinate.

These vectors can be written in the kronecker delta notation as

$$\underline{u}_k = (\delta_{k,1}, \delta_{k,2}, \delta_{k,3} \cdots, \delta_{k,n}) \quad k = 1, 2, 3 \cdots n$$

where $\delta_{k,j}$ is the kronecker delta defined by

$$\delta_{k,j} = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}$$

Note # If $\underline{x} = (x_1, x_2, \cdots, x_n) \in R^n$

Then

$$\underline{x} = x_1 \underline{u}_1 + x_2 \underline{u}_2 + \cdots + x_n \underline{u}_n$$

and this representation is unique.

Actually Collection $\{u_k : u_k = (\delta_{k,1} \cdots, \delta_{k,n}), k = 1, 2, \cdots n\}$ makes a basis for R^n

The Inner Product on R^n

The inner product of two vectors $\underline{x} = (x_1, x_2, \cdots x_n)$ and $\underline{y} = (y_1, y_2, y_3, \cdots, y_n)$ in R^n is

$$\underline{x} \cdot \underline{y} = \langle \underline{x} \ \underline{y} \rangle = \sum_{j=1}^{n} x_j y_j \longrightarrow ①$$

Notice that a function $\langle \cdot , \cdot \rangle : R^n \times R^n \longrightarrow R$ is called inner product function if following properties are true

i) # The inner product is additive in both its variables

i.e

$\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$

$\langle \underline{x}, \underline{y} + \underline{z} \rangle = \langle \underline{x}, \underline{y} \rangle + \langle \underline{x}, \underline{z} \rangle$

ii) # The inner product is symmetric: i.e

$$\langle \underline{x}\ \underline{y} \rangle = \langle \underline{y}\ \underline{x} \rangle$$

iii) # The inner product is homogeneous in both variables

i.e

$$\langle a\underline{x}, b\underline{y} \rangle = ab \langle \underline{x}\ \underline{y} \rangle$$

We check these properties for the inner product defined

(i) # $\langle \underline{x} + \underline{y}, \underline{z} \rangle = \sum_{i=1}^{n} (x_i + y_i) z_i$

$= \sum_{i=1}^{n} (x_i z_i + y_i z_i)$ distributive law

$= \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i$

$= \langle \underline{x}\ \underline{z} \rangle + \langle \underline{y}\ \underline{z} \rangle$

ii) #

$\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n} x_i y_i$

$= \sum_{i=1}^{n} y_i x_i$

$= \langle \underline{y}\ \underline{x} \rangle$

iii) $\langle a\underline{x}, b\underline{y} \rangle = \sum_{i=1}^{n} a x_i b y_i$

$= ab \sum_{i=1}^{n} x_i y_i$

$= ab \langle \underline{x}\ \underline{y} \rangle$

Absolute or Length or Norm

The Euclédean norm of a vector $\underline{x}$ in R^n is defined as

$$\|\underline{x}\| = \sqrt{\langle \underline{x}\ \underline{x}\rangle}$$

$$= \sqrt{\sum_{i=1}^{n} x_i^2}$$

This norm is generated by inner product.

Note # The distance between $\underline{x}$ & $\underline{y}$ is

$$|\underline{x} - \underline{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

The Cauchy-Schwarz Inequality

If $\underline{x}$ & $\underline{y}$ are two vectors in R^n, then

$$\|\langle \underline{x}\ \underline{y}\rangle\| \le \|\underline{x}\|\ \|\underline{y}\|$$

OR

$$\|\underline{x} \cdot \underline{y}\| \le \|\underline{x}\|\ \|\underline{y}\|$$

OR

If $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$ are real numbers, Then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \le \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

Proof # For t in R, form a vector $\underline{z} = t\underline{x} + \underline{y}$

Then

$$0 \le \|\underline{z}\|^2 = \langle \underline{z}, \underline{z}\rangle = \langle t\underline{x}+\underline{y},\ t\underline{x}+\underline{y}\rangle$$

$$= \langle t\underline{x}, t\underline{x}\rangle + \langle t\underline{x}\ \underline{y}\rangle + \langle \underline{y}\ t\underline{x}\rangle + \langle \underline{y}\ \underline{y}\rangle$$ additive property

$$= t^2\langle \underline{x}\ \underline{x}\rangle + t\langle \underline{x}\ \underline{y}\rangle + t\langle \underline{y}\ \underline{x}\rangle + \langle \underline{y}\ \underline{y}\rangle$$

$$= t^2\|x\|^2 + t\langle \underline{x}, \underline{y}\rangle + t\langle \underline{x}\ \underline{y}\rangle + \|\underline{y}\|^2$$

Symmetric property.

$$= \|\underline{x}\|t^2 + 2\langle \underline{x}\ \underline{y}\rangle t + \|\underline{y}\|^2$$

Let $\|\underline{x}\|^2 = A$ $\quad B = \langle \underline{x}\ \underline{y}\rangle$ $\quad C = \|\underline{y}\|^2$

Then

$$At^2 + 2Bt + C \geq 0 \quad \text{for all } t \text{ in } R$$

Now the equation

$$\varphi(t) = At^2 + 2Bt + C = 0$$

is quatratic in t and has roots t_1, t_2 say which may be

(i) Real and distinct

(ii) Real and equal

(iii) Complex conjugat.

If t_1, t_2 are real and distinct, then

$$\varphi(t) = A(t - t_1)(t - t_2)$$

is $-$ve for $t = \frac{t_1 + t_2}{2}$ $\quad \because \frac{t_1+t_2}{2}$ is such that $t_1 < \frac{t_1+t_2}{2} < t_2$ OR $t_2 < \frac{t_1+t_2}{2} < t_1$

But $\varphi(t) \geq 0 \quad \forall t \in R$.

Thus roots must be either equal or Complex Conjugate. i.e

$$\text{Disc} \leq 0$$

$$\Rightarrow 4B^2 - 4AC \leq 0$$

$$\Rightarrow B^2 \leq AC$$

$$\Rightarrow \|\langle \underline{x}\ \underline{y}\rangle\|^2 \leq \|\underline{x}\|^2 \|\underline{y}\|^2$$

$$\Rightarrow \|\langle \underline{x}\ \underline{y}\rangle\| \leq \|\underline{x}\|\ \|\underline{y}\|$$

OR

Let $\sum_{k=1}^{n} x_k^2 = A$ $\qquad \sum_{k=1}^{n} y_k^2 = B$

$$\sum_{k=1}^{n} x_k y_k = C$$

Now Consider

$$\sum_{k=1}^{n} (Bx_k - Cy_k)^2$$

$$= \sum_{k=1}^{n} \left(B^2x_k^2 + C^2y_k^2 - 2BC(x_k y_k)\right)$$

$$= B^2 \sum x_k^2 + C^2 \sum y_k^2 - 2BC \sum x_k y_k$$

$$= B^2 A + C^2 B - 2BC^2$$

$$= B^2 A - BC^2$$

Now

$$\sum_{k=1}^{n} (Bx_k - Cy_k)^2 \geq 0$$

$$\Rightarrow B^2A - BC^2 \geq 0$$

$$\Rightarrow B(AB - C^2) \geq 0$$

If $B = 0$, then inequality is trivial

If $B \neq 0$, then $B > 0$ and hence

$$AB - C^2 \geq 0$$

$$C^2 \leq AB$$

$$\sum_{k=1}^{n} x_k y_k \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k\right)$$

$$\|\underline{x} \cdot \underline{y}\|^2 \leq \|\underline{x}\|^2 \|\underline{y}\|^2$$

$$\Rightarrow \|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\| \, \|\underline{y}\|$$

Theorem# For vectors $\underline{x}, \underline{y}$ in R^n and c in R, the Euclidean norm has the following properties.

~~(i) Positive Definiteness $\|\underline{x}\| \geq 0$; $\|\underline{x}\| = 0$ iff $\underline{x} = \underline{0}$~~

(ii) Absolute Homogenerity $\|c\underline{x}\| = |c|\,\|\underline{x}\|$

(iii) Subadditivity $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

or triangular property

Proof# (i) $\|\underline{x}\| = (x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2} \geq 0$

Let $\|\underline{x}\| = 0$

$$\Rightarrow (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = 0$$

$$\Rightarrow x_1^2 + x_2^2 + \cdots + x_n^2 = 0$$

$$\Rightarrow x_1^2 = 0,\ x_2^2 = 0,\ \cdots\ x_n^2 = 0$$

$$\Rightarrow x_1 = 0 \quad x_2 = 0 \quad \cdots \quad x_n = 0$$

$$\Rightarrow \underline{x} = (0, 0, 0, \cdots 0) = \underline{0}$$

Conversely let $\underline{x} = \underline{0} = (0, 0, 0, \cdots 0)$

$$\|\underline{x}\| = (0^2 + 0^2 + 0^2 + \cdots + 0^2)^{1/2} = 0$$

$$\Rightarrow \|\underline{x}\| = 0$$

(ii) $c\underline{x} = (cx_1, cx_2, cx_3, \cdots, cx_n)$

$$\|c\underline{x}\| = (c^2x_1^2 + c^2x_2^2 + \cdots + c^2x_n^2)^{1/2}$$

$$= |c|\,(x_1^2, x_2^2, \cdots, x_n^2)^{1/2}$$

$$= |c|\,\|\underline{x}\|^2$$

OR

$$\|c\underline{x}\|^2 = \langle c\underline{x}\ \ c\underline{x}\rangle = c^2\langle \underline{x}\ \ \underline{x}\rangle$$

$$= c^2\|\underline{x}\|^2$$

$$\Rightarrow \|c\underline{x}\| = |c|\,\|\underline{x}\|$$

Theorem# Let $\underline{x}, \underline{y} \in R^n$. Then

(i) $\|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$ (Pythagorian Theorem) iff $\underline{x}\cdot\underline{y} = 0$.

(ii) $\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2$ (parallelogram law)

(iii) $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ Triangle law

Proof# (i) $\|\underline{x}+\underline{y}\|^2 = (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y})$

$$= \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y}$$

$$= \|\underline{x}\|^2 + \underline{x}\cdot\underline{y} + \underline{x}\cdot\underline{y} + \|\underline{y}\|^2$$

$$= \|\underline{x}\|^2 + 2\,\underline{x}\cdot\underline{y} + \|\underline{y}\|^2.$$

Thus equality $\|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$ iff $\underline{x}\cdot\underline{y} = 0$

(ii)

$$\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2$$

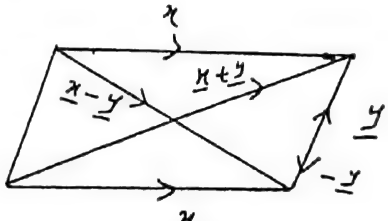
$$= (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y}) + (\underline{x}-\underline{y})\cdot(\underline{x}-\underline{y})$$

$$= \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y} + \underline{x}\cdot\underline{x} - \underline{x}\cdot\underline{y} - \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y}$$

$$= \|\underline{x}\|^2 + 2\,\underline{x}\cdot\underline{y} + \|\underline{y}\|^2 + \|\underline{x}\|^2 - 2\,\underline{x}\cdot\underline{y} + \|\underline{y}\|^2$$

$$= 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2$$

Geometrically it means that the sum of the squares of the diagonals of a parallelogram with sides $\underline{x}$ & $\underline{y}$ is equal to the double the sum of the squares of the sides.



(iii) $\|\underline{x}+\underline{y}\|^2 = \langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle$

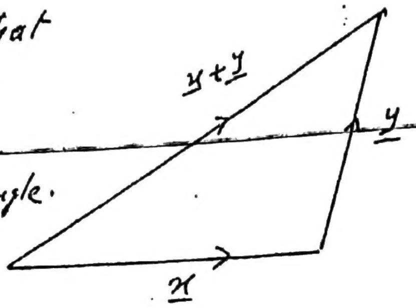
$$= \|\underline{x}\|^2 + 2\,\underline{x}\cdot\underline{y} + \|\underline{y}\|^2$$

$$\leq \|\underline{x}\|^2 + 2\|\underline{x}\|\|\underline{y}\| + \|\underline{y}\|^2 \quad \text{Cauchy Schwarz Inequality}$$

$$= (\|\underline{x}\| + \|\underline{y}\|)^2$$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

Geometrically it means that the length of a side of any triangle is less than the sum of the other sides of the triangle.
other two t



Minkoskis Inequality for Real Numbers

Let $a_1, a_2 \dots a_n$ and $b_1, b_2, b_3 \dots b_n$ be any real numbers. Then

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

Proof #

$$\left[\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right]^2 = \sum_{i=1}^{n} a_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2$$

$$\geq \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \quad \text{By Schwarz's Inequality}$$

$$\geq \sum_{i=1}^{n} (a_i^2 + 2a_i b_i + b_i^2)$$

$$\geq \sum_{i=1}^{n} (a_i+b_i)^2$$

$$\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2} \geq \sqrt{\sum_{i=1}^{n}(a_i+b_i)^2}$$

or

$$\sqrt{\sum (a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

(Proved)

Schwarz Inequality for Complex Numbers

If $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ are complex numbers, then

$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2$$

Proof# Let $A = \sum |a_i|^2 \quad B = \sum |b_i|^2 \quad C = \sum a_i \bar{b}_i$

$$\sum |Ba_i - Cb_i|^2 = \sum (Ba_i - Cb_i)\overline{(Ba_i - Cb_i)}$$

$$= \sum (Ba_i - Cb_i)(B\bar{a}_i - \bar{C}\bar{b}_i)$$

$$= B^2 \sum |a_i|^2 - B\bar{C}\sum a_i \bar{b}_i - BC\sum \bar{a}_i b_i + |C|^2 \sum |b_i|^2$$

$$= B^2 \sum |a_i|^2 - B\bar{C}C - BC\bar{C} + |C|^2 \sum |b_i|^2$$

$$= B^2 A - B|C|^2 - B|C|^2 + |C|^2 B$$

$$= B^2 A - B|C|^2$$

$$= B(BA - |C|^2)$$

$\because$ Sum of squares is never negative.

$$\therefore \quad \sum |Ba_i - Cb_i|^2 \geq 0$$

$$\Rightarrow \quad B(BA - |C|^2) \geq 0$$

If $B = 0$, Then $b_1 = 0 = b_2 = b_3 = \cdots = b_n$ and the equality holds. Let $B > 0$

$$\Rightarrow \quad BA - |C|^2 \geq 0$$

$$BA \geq |C|^2$$

$$\Rightarrow \quad |C|^2 \leq AB$$

$$\left|\sum a_i \bar{b}_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \text{ (Proved)}$$

Theorem # If $x, y, z \in R^n$, then

(a) $\|x-y\| \geq \big|\|x\| - \|y\|\big|$

(b) $\|x-y\| \leq \|x-z\| + \|z-y\|$

~~(c) $x \cdot y \leq \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$~~

(d) $\|x\| = \|y\|$ iff $(x-y)\cdot(x+y) = 0$

Proof # (a) $\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$

$$\Rightarrow \|x\| - \|y\| \leq \|x-y\| \longrightarrow (1)$$

Again

$$\|y\| = \|(y-x)+x\| \leq \|y-x\| + \|x\|$$
$$= \|x-y\| + \|x\|$$

$$\|y\| - \|x\| \leq \|x-y\|$$

$$\Rightarrow -[\|x\| - \|y\|] \leq \|x-y\| \longrightarrow (2)$$

By (1) & (2)

$$\big|\|x\| - \|y\|\big| \leq \|x-y\|$$

(b): $\|x-y\| = \|x-z+z-y\|$
$$\leq \|x-z\| + \|z-y\|$$

(c): $\|x+y\|^2 - \|x-y\|^2$

$$= (x+y)\cdot(x+y) - (x-y)\cdot(x-y)$$
$$= \|x\|^2 + 2x\cdot y + \|y\|^2 - \{\|x\|^2 - 2x\cdot y + \|y\|^2\}$$
$$= 4x\cdot y$$

$$\Rightarrow x\cdot y = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$$

(d) Let $\|x\| = \|y\| \Rightarrow \|x\|^2 = \|y\|^2$

$$\Leftrightarrow x\cdot x = y\cdot y$$
$$\Leftrightarrow x\cdot x - y\cdot y = 0$$
$$\Leftrightarrow x\cdot x - y\cdot y + x\cdot y - x\cdot y = 0$$
$$\Leftrightarrow x\cdot(x-y) + y\cdot(x-y) = 0$$
$$\Leftrightarrow (x-y)\cdot(x+y) = 0 \quad \text{Proved.}$$

Norms on R^n

A norm on R^n is any function n from R^n to R ~~that is positive, Absolute homogeneous and sub additive.~~

There are many norms on R^n. All are interesting. Several are also useful.

The Eucledean norm $\|\underline{x}\| = \sqrt{\langle x, x\rangle}$ is the norm generated by inner product and is a special norm

For $\underline{x} = (x_1, x_2) \in R^2$, define.

$$\|\underline{x}\|_1 = |x_1| + |x_2|$$

Then $\|\cdot\|_1$ is a norm on R^2

For $\underline{x} = (x_1, x_2)$ in R^2, define.

$$\|\underline{x}\|_\infty = \max\{|x_1|, |x_2|\}$$

Then $\|\cdot\|_\infty$ is a norm on R^2.